Complex multiplication values of modular forms

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Contents

1 Complex multiplication

Lecture 1 (1 hour)
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Consider an elliptic curve E over $\mathbb{C}$; its endomorphisms ring $\mathrm{End}(E)$ is known to be $\mathbb{Z}$ in general, or some ring $\mathcal{O} \subseteq K := \mathbb{Q}(\sqrt{d})$ with $\operatorname{disc} d < 0$. In the latter case, E is said to have a complex multiplication by $\mathcal{O}$.

Since we are over $\mathbb{C}$, we can identify E with a complex torus, i.e., there is a lattice $\Lambda \subseteq \mathbb{C}$ such that $E \cong E_\Lambda$, where E_Λ is the elliptic curve associated to the lattice. We set $E_\Lambda := \mathbb{C}/\Lambda$, where the class of the point z in $\mathbb{C}/\Lambda$ corresponds to the point in the plane $(p(z,\Lambda), p'(z,\Lambda))$; equivalently, E is given in $\mathbb{P}^2$ by the equation $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$.

We have that $\mathrm{End}(E_\Lambda) = \{\, \alpha \in \mathbb{C} \mid \alpha\Lambda \subseteq \Lambda \,\}$. Moreover, if two lattices are homotetic (obtained one from the other by the multiplication by a complex number) then the two elliptic curves obtained are isomorphic.

1.1 proposition. *Let $0 \neq \lambda \in \Lambda$; then E_Λ has complex multiplication by $\mathcal{O}_K$ if and only if $1/\lambda\Lambda \subseteq K$ is a fractional ideal.*

We obtain a bijective map from the set of ideal classes $\mathrm{Cl}(K)$ of K to $\mathrm{Ell}(K)$, the set of elliptic curves with complex multiplication by $\mathcal{O}_K$, modulo isomorphisms. In particular, K acts on $\mathrm{Ell}(K)$ by $[\xi] \cdot [E_\Lambda] := [E_{\xi^{-1}\Lambda}]$.

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1.2 example. Let $\Lambda \simeq \mathbb{Z}[\iota] \subseteq \mathbb{C}$ and let $K := \mathbb{Q}(\iota)$. For the elliptic curve $\mathbb{C}/\Lambda \cong (y^2 = x^3 + x)$, the action is given by $\iota \cdot (x, y) = (-x, \iota y)$.

1.3 proposition.

1. *Let E be a complex elliptic curve, and $\sigma \in \mathrm{Aut}(\mathbb{C})$; then $\mathrm{End}(E^\sigma) = \mathrm{End}(E)$.*
2. *If E has a complex multiplication by $\mathcal{O}_K$, then it is defined over $\overline{\mathbb{Q}}$ and $j(E) \in \overline{\mathbb{Q}}$.*

Moreover, the ideal class group is finite, so, for example, $\{j(E)^\sigma\}$ is finite.

1.4 theorem. *Let E be a complex elliptic curve with complex multiplication by $\mathcal{O}_K$; then:*

1. *$j(E)$ is an algebraic integer;*
2. *$K(j(E)) = H$, the Hilbert class field of K;*
3. *the action of $\mathrm{Gal}(H/K)$ on $j(E) = j(E_{\tilde{\varsigma}})$ is given by the Artin map of class field theory,*

$$(\bullet, H/K)\colon \mathrm{Cl}(K) \to \mathrm{Gal}(H/K),$$

defined by $j(E_{\tilde{\varsigma}})^{(p,H/K)} = j(E_{p^{-1}\tilde{\varsigma}})$.

Now we know that $j(E)$ is an algebraic integer when E has a complex multiplication by $\mathcal{O}_K$. Can we describe it more explicitly? This question was explored already in the beginning of the twentieth century, for example by Berwich and Deuring. In the eighties, Gross and Zagier proved an explicit formula for the norm of $j(E)$.

Gross and Zagier considered j as a function from the moduli space of elliptic curves to the affine line, that gives an identification of the two. But j is not canonical, for example, $j + 1$ has the same properties. So, Gross and Zagier studied the differences between two of these j-invariants.

Consider d_1, d_2 with negative discriminants, and let $K_i := \mathbb{Q}(\sqrt{d_i})$; then they defined

$$\mathcal{J}(d_1, d_2) = \prod_{[E_i] \in \mathrm{Ell}(K_i)} (j(E_1) - j(E_2)).$$

This is more canonical since for example, taking j or $j + 1$ does not change this function. Note that if $d_2 = -3$, then $j(E_2) = 0$, so as a special case we recover the j-invariants we started with (more precisely, $\mathcal{J}(d_1, -3) = N_{E_1/K_1}(j(E_1))$. Starting with these observations, they proved the following.

1.5 theorem (Gross-Zagier). *We have*

$$\mathcal{J}(d_1, d_2)^2 = \pm \prod_{\substack{x \in \mathbb{Z} \\ n, n' \in \mathbb{Z}_{>0} \\ x^2 - 4nn' = d_1 d_2}} n^{\varepsilon(n')},$$

where $\varepsilon(n') \in \{\pm 1\}$ is the value of some "genus character".

In particular, note that $n \leq 1/4 d_1 d_2$.

1.6 COROLLARY. *If a prime p divides $\mathcal{J}(d_1, d_2)$, then $p \mid 1/4(d_1 d_2 - x^2)$ for some $x \in \mathbb{Z}$, with $|x| < d_1 d_2$ and $x \equiv \sqrt{d_1 d_2}\ (4)$.*

Proof. Let $L := H_1 \cdot H_2$ be the composition of the two Hilbert classes. If $p \mid \mathcal{J}(d_1, d_2)$, then there must be a prime ideal $P \subseteq \mathcal{O}_L$ over p, and two elliptic curves $E_i \subseteq \mathrm{Ell}(K_i)$ such that $P \mid (j(E_1) - j(E_2))$. This implies that $\mathcal{O}_L/P$ is an extension of $\mathbb{F}_p$ and moreover $E_1 \cong E_2$ over $\overline{\mathcal{O}_L/P} = \overline{\mathbb{F}}_p$.

Hence, $\mathrm{End}(E_i/\mathbb{F}_p)$ contains $\mathcal{O}_{K_1}$ and $\mathcal{O}_{K_2}$ with $K_1 \neq K_2$ and $(d_1, d_2) = 1$. By the work of Deuring, $\mathrm{End}(E_i/\mathbb{F}_p) \cong \mathcal{O}_B \subseteq B := B_{p,\infty}$, where B is the quaternion algebra ramified at p, ∞.

Putting the two together, we obtain that $\mathcal{O}_B$ contains some α_i with $\alpha_i^2 = d_i$, therefore $\alpha := \alpha_1 \alpha_1 \in \mathcal{O}_B$. Now, define $x := \mathrm{tr}(\alpha) \in \mathbb{Z}$; by the properties of B, we have that $x^2 < N(\alpha)$ (because B is ramified at ∞) and $x^2 \equiv d_1 d_2\ (p)$ (because B is ramified at p). Working with $d_i/2$ or $d_i + 1/2$ (depending on the parity), we could have obtained also the factor $1/4$. $\square$

We know that in the case of $\mathbb{C}$, we can parametrize elliptic curves modulo isomorphisms with the upper half complex plane $\mathbb{H}$, modulo $\mathrm{SL}_2(\mathbb{Z})$. So we can study the points in the upper half plane that corresponds to elliptic curves with complex multiplication. These turn out to be (sloppily) $\tau \in \mathbb{H} \cap K$, or, more precisely, points τ such that $a\tau^2 + b\tau + c = 0$ with a, b, c such that $b^2 - 4ac = d$.

Of course we can reinterpret the function j as a function coming from $\mathbb{H}$. Mimicking what we did before, we consider $j(z_1) - j(z_2)$, where z_i are point in $\mathbb{H}$. Zagier also gave a proof that $\log |j(z_1) - j(z_2)|$ is the resolvent kernel for the (regularized) hyperbolic Laplacian.

2 Complex multiplication values of Borcherds products

Lecture 2 (1 hour)
October 5th, 2010

Let $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$; we already said that $Y(1) := \mathbb{H}/\Gamma(1)$ is in a bijection with the set of elliptic curves over $\mathbb{C}$; we have also the map j from this set to $\mathbb{C}$. Hence, we can view also $j \colon \mathbb{H} \to \mathbb{C}$, a $\Gamma(1)$-invariant functions on the complex upper half plane. In other words,

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau), \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1).$$

Now, we consider instead functions f such that $f(\gamma\tau) = (c\tau + d)^k f(z)$ for every $\gamma \in \Gamma(1)$. These are called *modular forms* of weight k for $\Gamma(1)$. In particular, j is a modular form of weight 0. So, the Gross-Zagier function $j(\tau_1) - j(\tau_2)$ is defined on $Y(1) \times Y(1)$.

2.1 EXAMPLE. Let $V := M_2(\mathbb{Q})$, $Q = \det$ be the determinant as a quadratic form (with signature $(2, 2)$; $\mathrm{SL}_2 \times \mathrm{SL}_2$ acts on V by $(\gamma_1, \gamma_2) \cdot X := \gamma_1 X \gamma_2^{-1}$, and this action leaves the quadratic form Q unchanged, so we have a map $\mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}(V)$.

Indeed, we can construct modular forms using the so called "theta lift".

2.1 *Setup*

Let (V, Q) be a quadratic space of signature $(n, 2)$. Consider $\mathrm{SO}(V)$ and $H := \mathrm{GSpin}(V)$. This sits in an extension

$$1 \to \mathbb{G}_m \to H \to \mathrm{SO}(V) \to 1.$$

Consider $L \subseteq V$ an even unimodular lattice (for simplicity), and $K \subseteq H(\hat{\mathbb{Q}})$ a compact open subgroup.

We consider also a space $\mathbb{D}$ that is for us a generalization of $\mathbb{H}$; it is realized explicitly as the subset of the Grassmannian of $V(\mathbb{R})$ determined by the hyperplanes Z of dimension 2 such that $Q|_Z < 0$.

Finally, define the *Shimura variety* as $X_{V,K} = X_K := {}_{H(\mathbb{Q})}\backslash^{(\mathbb{D}\times H(\hat{\mathbb{Q}}))}/_K$. This is a quasi-projective variety over $\mathbb{Q}$, of dimension n.

2.2 example. If $n = 2$, $V = M_2(\mathbb{Q})$, $Q = \det$, then H is the group of pairs $(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2$ with $\det g_1 = \det g_2$. We can take K as the subgroup of pairs in H such that both elements are in $\mathrm{GL}_2(\hat{\mathbb{Z}})$. Finally, we get $X_K \cong Y(1) \times Y(1)$.

2.2 *Special cycles*

Let us consider the case of signature $(n, 2)$. Let $W \subseteq V$ be positive definite, with dimension $r < n$, defined over $\mathbb{Q}$; then $W^\perp \subseteq V$ has signature $(n-r, 2)$, and $H_W := \mathrm{GSpin}(W^\perp)$ embeds in H. Moreover, we have a map $X_{W^\perp, K\cap H_W(\hat{\mathbb{Q}})} \to X_K = X_{V,K}$, and its image defines a cycle $Z(W)$ of codimension r.

2.3 example. Take $x_0 \in L \subseteq V$, with V primitive and $Q(x_0) = m > 0$. Then $W = \mathbb{Q}x_0$ and $Z(m) := Z(\mathbb{Q}x_0)$. This is called the discriminant m.

2.4 example. If $V = M_2(\mathbb{Q})$ and $X_K = Y(1) \times Y(1)$, then $Z(m) = \Gamma(m)$, the m-th Hecke correspondence on $Y(1)$.

2.3 *The Siegel theta function*

Let M be a positive definite lattice of rank r; let $\Theta_M(\tau) := \sum_{\lambda \in m} e^{2\pi\imath Q(\lambda)\tau}$ with $\tau \in \mathbb{H}$. Using the Poisson summation formula, one can prove that this is a modular form of rank $r/2$ for $\Gamma \subseteq \Gamma(1)$.

The Siegel theta function is defined to be

$$\Theta(\tau, z, h) := v \sum_{d \in h\cdot L} e(Q(\lambda_{z\perp})\tau + Q(\lambda_2)\overline{\tau}).$$

The following are some of its properties:

1. $\Theta(\gamma'\tau, z, h) = (c\tau + d)^{n+2/2}\Theta(\tau, z, h)$ for every $\gamma' \in \Gamma(1)$;
2. $\Theta(\tau, \gamma z, \gamma h) = \Theta(\tau, z, h)$ for every $\gamma \in H(\mathbb{Q})$;

3. $\Theta(\tau, z, kh) = \Theta(\tau, z, h)$ for every $k \in K$.

2.5 definition. Let $f \in M^!_{1-n/2}$ (a weakly holomorphic modular form of weight $1 - n/2$ for $\Gamma(1)$). Then the *theta integral* is

$$\varphi(z, h, f) := \int\limits_{\Gamma(1)\backslash\mathbb{H}}^{\text{reg}} f(\tau)\Theta(\tau, z, h)\,\mathrm{d}\,\mu(\tau).$$

The regularization is needed to ensure that the integral converges: instead of integrating over a fundamental domain F, we consider the limit for $T \to \infty$ of the integral over the fundamental domain truncated at $\Im\tau \leq T$.

2.6 theorem (Borcherds). *Let $f(\tau) = \sum_{m \geq m_0} c(m)q^m \in M^!_{1-n/2}$, with $m < 0 \Rightarrow c(m) \in \mathbb{Z}$. Then there is a meromorphic modular form $\psi(z, h, f)$ on X_K such that:*

1. *ψ has weight $c(0)/2$;*
2. *the divisor associated to ψ is $Z(f) = \sum_{m>0} c(-m)Z(m)$;*
3. *product expansion;*
4. $\log \|\psi(z, h, f)\| = -1/4\varphi(z, h, f)$.

2.7 example. Consider again $V = M_2(\mathbb{Q})$, $X_K = Y(1) \times Y(1)$; take $f = j(\tau) - 744 =: \mathcal{J}$; then $\mathcal{J}(\tau) = q^{-1} + 0 + \cdots \in M^!_0$, and

$$\Psi(z, \mathcal{J}) = q_1^{-1} \prod_{\substack{m,n \in \mathbb{Z} \\ m>0}} (1 - q_1^m q_2^n)^{c(mn)} = j(z_1) - j(z_2).$$

Moreover, we have $Z(\mathcal{J}) = Z(1) = \Gamma(1)$.

Lecture 3 (1 hour)
October 6th, 2010

The goal is to evaluate the function Ψ at complex multiplication cycles $Z(W)$, that is, obtain the rational number

$$\Psi(Z(W), f) = \prod_{[z,h] \in Z(W)} \Psi(z, h, f).$$

So, if we take the logarithm, we obtain

$$\log \|\Psi(Z(W), f)\| = -1/4 \sum \varphi(z, h, f).$$

The question is what is $\varphi(Z(W), f)$.

Consider a lattice $L \subseteq V$, then $P := L \cap W$ is a positive definite lattice of dimension n; associate to any positive definite lattice we have a theta series $\Theta_P(\tau) = \sum_{\lambda \in P} e^{2\pi i Q(\lambda)\tau} \in M_{n/2}$. Now, let $N := L \cap W^\perp$; it is negative definite of dimension 2, and Θ_N gives a genus theta series $\Theta_{[n]}$, that is related (by Siegel-Weil) to the Eisenstein series $E_N(\tau, s, 1)$ of weight 1. It turns out this has a functional equation, that relates the values of E_N for s and $-s$, and in particular, $E_N(\tau, 0, 1) = 0$, and it makes sense to study the derivative $\mathcal{E}_N(\tau) = \frac{\mathrm{d}}{\mathrm{d}s} E_N(\tau, s, 1)|_{s=0}$.

2.8 theorem (Schofer). *Let $f \in M^!_{1-n/2}$ and $W \subseteq V$; if $f = \sum_{m \geq m_0} c(m) q^m$ (where $q = e^{2\pi i \tau}$), then*

$$\varphi(Z(W), f) = \deg Z(W) \cdot \sum_{m>0} c(-m) \cdot d(m),$$

where $d(m)$ is the m-th coefficient of $\Theta_P \cdot \mathcal{E}_N$.

This theorem gives another proof of Gross-Zagier's formula.

3 Traces of singular moduli

Let E be a complex elliptic curve with complex multiplication by $\mathcal{O}_K \subseteq K = \mathbb{Q}(\sqrt{d})$, with $d > 0$. Then Gross-Zagier formula gives the norm of the j-invariant, $\mathrm{N}_{H/K}(j(E)) \in \mathbb{Z}$. We want to understand also the trace of the j-invariant. The trace is

$$\mathrm{tr}_{H/K}(j(E)) = \sum_{[\xi] \in \mathrm{Cl}(K)} j(E_\xi) = \sum_{Q \in \mathcal{Q}_d / \Gamma(1)} j(\tau_Q),$$

where $\mathcal{Q}_d$ is the set of all primitive quadratic forms of discriminant $-d$, i.e. triples $(a, b, c) \in \mathbb{Z}^3$ such that $b^2 - 4ac = -d$.

In order to proceed, we have to do some modifications; the first is to replace j by $\mathcal{J} := j - 744$; the second is to use the weight $2/\omega_Q$, with $\omega_Q = |\Gamma(1)|_Q$, that is 2, 4, or 6 depending on the number of roots of unity in $\mathrm{Cl}(K)$; the third is to sum over all quadratic forms, not only over the primitive one.

3.1 definition. We define

$$g(\tau) := \Theta_1(\tau) \cdot \frac{E_4(4\tau)}{\eta(4\tau)^6} \in M^!_{3/2}(\Gamma_0(4))^+,$$

where:

$$\Theta_1(\tau) = \sum_{m \in \mathbb{Z}} (-1)^n q^{n^2}$$
$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n$$
$$\eta(\tau) = q^{1/24} \prod_{n \geq 0} (1 - q^n)$$

3.2 theorem (Zagier). *We have*

$$g(E) = q^{-1} - 2 - \sum_{\substack{d>0 \\ d \equiv 0,3 \ (4)}} t_d(\mathcal{J}) q^d,$$

where

$$t_d(\mathcal{J}) = \sum_{Q \in \mathcal{Q}_d / \Gamma(1)} 2/\omega_Q \mathcal{J}(\tau_Q).$$