3
Cone Theorems

In Chapter 1 we proved the Cone Theorem for smooth projective varieties, and we noted that the proof given there did not work for singular varieties. For the minimal model program certain singularities are unavoidable and it is essential to have the Cone Theorem for pairs (X, Δ). Technically and historically this is a rather involved proof, developed by several authors. The main contributions are [Kaw84a, Rei83c, Sho85].

Section 1 states the four main steps of the proof and explains the basic ideas behind it. There is a common thread running through all four parts, called the basepoint-freeness method. This technique appears transparently in the proof of the Basepoint-free Theorem. For this reason in section 2 we present the proof of the Basepoint-free Theorem, though logically this should be the second step of the proof.

The remaining three steps are treated in the next three sections, the proof of the Rationality Theorem being the most involved.

In section 6 we state and explain the relative versions of the Basepoint-free Theorem and the Cone Theorem.

With these results at our disposal, we are ready to formulate in a precise way the log minimal model program. This is done in section 7. In dimension two the program does not involve flips, and so we are able to treat this case completely.

In section 7 we study minimal models of pairs. It turns out that this concept is not a straightforward generalization of the minimal models of smooth varieties 2.13). The definitions are given in 3.50) and their basic properties are described in 3.52).

3.1 Introduction to the Proof of the Cone Theorem

In section 1.3, we proved the Cone Theorem for smooth varieties. We now begin a sequence of theorems leading to the proof of the Cone Theorem in the general case. This proof is built on a very different set of ideas. Applied even in the smooth case, it gives results not accessible by the previous method; namely it proves that extremal rays can always be contracted. On the other hand, it gives little information about the curves that span an extremal ray. Also, this proof works only in characteristic 0. Before proceeding, we reformulate slightly the Vanishing Theorem 2.64):

3.1 Theorem.

Data	Hypothesis	Thesis
• *Y smooth complex projective variety* • *$\sum d_i D_i$ $\mathbb{Q}$-divisor* • *L line bundle*	• *$D := L + \sum d_i D_i$ is nef and big* • *$\sum D_i$ is snc*	$H^i(Y, \mathcal{O}_Y(K_Y + \lceil D \rceil)) = 0$ *for $i > 0$*

3.2. We prove four basic theorems finishing with the Cone Theorem. The proofs of these four theorems are fairly intervowen in history. For smooth threefolds [Mor82] obtaines some special cases. The first general result for threefolds was obtained by [Kaw84b], and completed by [Ben83] and [Rei83c]. Nonvanishing was done by [Sho85]. The Cone Theorem appears in [Kaw84a] and is completed in [Kol84]. See [KMM87] for a detailed treatment and for generalizations to the relative case.

3.3 Theorem (Basepoint-free Theorem).

Data	Hypothesis	Thesis
• *(X, Δ) proper klt pair* • *Δ effective* • *D nef Cartier divisor*	*$\exists a > 0$: $aD - K_X - \Delta$ is nef and big*	*$\lvert bD \rvert$ is basepoint-free for $b \gg 0$*

3.4 Theorem (Non-vanishing Theorem).

Data	Hypothesis	Thesis
• *X proper variety* • *D nef Cartier divisor* • *G a $\mathbb{Q}$-divisor*	• *$\exists a > 0$: $aD + G - K_X$ is $\mathbb{Q}$-Cartier, nef and big* • *$(X, -G)$ is klt*	$H^0(X, mD + \lceil G \rceil) \neq 0$ *for $m \gg 0$*

3.5 Theorem (Rationality Theorem).

Data	Hypothesis
• *(X, Δ) proper klt pair* • *Δ effective* • *H Cartier, nef and big* • *a positive integer*	• *$K_X + \Delta$ not nef* • *$a(K_X + \Delta)$ Cartier*

Thesis

- $r(H) := \max\{t \in \mathbb{R} \mid \lvert H + t(K_X + \Delta)$ *not nef*$\} \in \mathbb{Q}$
- $r(H) = u/v$ *with $u, v \in \mathbb{Z}$ and $0 < v < a(\dim X + 1)$*

3.6 Complement. Notation as above. *Then there is an extremal ray R such that $R \cdot (K_X + \Delta) < 0$ and $R \cdot (H + r(K_X + \Delta)) = 0$.*

3.7 Theorem (Cone Theorem).

Data	Thesis
• (X, Δ) *projective klt pair* • Δ *effective*	• $\exists$ *countably many curves* $C_j \subseteq X$ *with* $0 < -(K_X + \Delta) \cdot C_j \leq 2 \dim X$ • $\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta)\geq 0} + \sum \mathbb{R}_+[C_j]$

Data	Thesis
• *H ample $\mathbb{Q}$-divisor* • $\varepsilon \in \mathbb{R}^+$	$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{(K_X+\Delta+\varepsilon H)\geq 0} + \sum_{\mathit{finite}} \mathbb{R}_+[C_j]$

Data	Thesis
$F \subseteq \overline{\mathrm{NE}}(X)$ *a* $(K_X + \Delta)$*-negative extremal face*	$\exists!\, \mathrm{cont}_F \colon X \to Z$ *morphism (the* contraction*) to a projective variety such that* $(\mathrm{cont}_F)_\star \mathscr{O}_X = \mathscr{O}_Z$ *and* $C \subseteq X$ *is contracted iff* $[C] \in F$

Data	Hypothesis	Thesis
L line bundle on X	$\forall C \colon [C] \in F, L \cdot C = 0$	$\exists L_Z$ *line bundle on* Z *such that* $L \cong \mathrm{cont}_F^\star L_Z$