1 Introduction

Let $E_\Lambda$ be the elliptic curve associated to lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, oriented in the sense that $\Im(\omega_1/\omega_2) > 0$. We know that $E_{\Lambda_1} \cong E_{\Lambda_2}$ if and only if $\Lambda_1 = a\Lambda_2$ for some $a \in \mathbb{C} \setminus \{0\}$.

1.1 Definition. A modular function is a function $M_{1,1} \to \mathbb{C}$ where $M_{1,1}$ is the space of elliptic curves over $\mathbb{C}$.

A modular function can be viewed as a function

$$F : \{\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2\} \to \mathbb{C}$$

with the property that $F(a\Lambda) = F(\Lambda)$ for all lattice $\Lambda$ and $a \in \mathbb{C} \setminus \{0\}$.

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1.2 Definition. A modular form of weight $k$, with $k \in \mathbb{Z}$, is a modular function $F$ such that

$$(1) \quad F(a\Lambda) = a^{-k}F(\Lambda).$$

Let $SL(2,\mathbb{Z})$ the set of integral matrices with determinant 1; from now on, we will denote an element of $SL(2,\mathbb{Z})$ with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\Lambda := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ a lattice; if $A \in SL(2,\mathbb{Z})$, $A(\omega_1,\omega_2)$ is just another basis for the same lattice. Let $\mathcal{H} := \{ \tau \in \mathbb{C} \mid \Im \tau > 0 \}$ be the half complex plane with positive imaginary part. The matrix group $SL(2,\mathbb{Z})$ acts on $\mathcal{H}$ by $A(\tau) := \frac{a\tau + b}{c\tau + d}$.

Now let $F$ be a modular form of weight $k$; we can associate to it a function $f : \mathcal{H} \to \mathbb{C}$ by $f(\tau) := F(\mathbb{Z}\tau \oplus \mathbb{Z})$; in this context, condition (1) ensure that

$$f(A(\tau)) = F \left( \mathbb{Z} \frac{a\tau + b}{c\tau + d} \oplus \mathbb{Z} \right) = F \left( \frac{1}{c\tau + d}(\mathbb{Z}(a\tau + b) \oplus \mathbb{Z}(c\tau + d)) \right) = (c\tau + d)^k F(\mathbb{Z}\tau \oplus \mathbb{Z}) = (c\tau + d)^k f(\tau).$$

Conversely, if we have $f : \mathcal{H} \to \mathbb{C}$ such that $f(A(\tau)) = (c\tau + d)^k f(\tau)$, then we can associate to it a function $F$ as in $F(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$; this is a modular form. In particular one obtain that these correspondences are each the inverse of the other. So we can give the following equivalent definition.

1.3 Definition. A modular form of weight $k$, with $k \in \mathbb{Z}$, is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying (2), and not growing too fast as $\tau \to \infty$.

The last condition will ensure later that modular forms corresponds to sections of a line bundle on $\overline{\mathcal{M}}_{1,1}$. Another way to say the same thing is to define for every $f : \mathcal{H} \to \mathbb{C}$, $k \in \mathbb{Z}$, and $A \in SL(2,\mathbb{Z})$ the function $f|_{k,A} : \mathcal{H} \to \mathbb{C}$ with $f|_{k,A}(\tau) := (c\tau + d)^{-k} f(A\tau)$; then we request $f = f|_{k,A}$ for every $A$.

Why modular forms are useful in mathematics?

1. There are very few modular forms; the space of modular forms of weight $k$ is a vector space of finite dimension.
2. They occur naturally in many fields of mathematics and physics.

2 The modular group

Consider the previously defined action of $SL(2,\mathbb{Z})$ on $\mathcal{H}$; since $-I$ acts trivially, we can also say that $\Gamma := SL(2,\mathbb{Z})/\{\pm I\}$ acts on $\mathcal{H}$. We define two special elements:

1. $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, such that $S\tau = -\tau^{-1}$;
2. $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, such that $T\tau = \tau + 1$.

Moreover, we have $S^2 = I = (ST)^3$ in $\Gamma$. 

2
Now we can find a fundamental domain for the action of $\Gamma$:

$$D := \{ \tau \in \mathbb{H} \mid |\tau| \geq 1, -1/2 \leq \Re \tau \leq 1/2 \}.$$ 

We define $q := e^{2\pi i/3}$, so that $-\bar{q} = q^2 = e^{4\pi i/3}$. Pictorially, we have Figure 1.

2.1 PROPOSITION. The so defined $D$ is a fundamental domain for the action of $\Gamma$ on $\mathbb{H}$; in particular we have that:

1. every point in $\mathbb{H}$ has a conjugate point in $D$ with respect to the action;
2. if $\tau, \tau' \in D$ are conjugate and different, then $\Re \tau = \pm 1/2$ and $\tau' = \tau \pm 1$, or $|\tau| = 1, \tau' = -1/\tau$;
3. let $\tau \in D$ and $I(\tau) := |\{ A \in \Gamma \mid A \tau = \tau \}|$; then $I(\tau) = 1$ unless $\tau = i$ ($I(i) = 2$) or $\tau \in \{ q, -\bar{q} \}$ ($I(q) = I(-\bar{q}) = 3$).

We can translate this situation into the stack language; the closure of $\mathbb{H}/\Gamma$ can be identified with the weighted projective space $\mathbb{P}(4,6)$; so we can define a generic $1/2 : 1$ map to $\mathbb{P}^1$ (with two special point, corresponding to $i$ and $q$).

2.2 DEFINITION. Let $k \in \mathbb{Z}$; a weakly modular form of weight $k$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that $f(A \tau) = (c \tau + d)^k f(\tau)$ for every $A \in \text{SL}(2, \mathbb{Z})$.

Let $L$ be the space of meromorphic function $f : \mathbb{H} \to \mathbb{C}$; $\text{SL}(2, \mathbb{Z})$ acts on $L$ in this way:

2.3 COROLLARY. A function $f$ is weakly modular of weight $k$ if and only if $f(-\tau^{-1}) = f(\tau)$ and $f(\tau + 1) = f(\tau)$ (e.g. if it is invariant with respect to $S$ and $T$).

Given $\tau$, write $q := e^{2\pi i \tau}$; consider a weakly modular form of weight $k$; then we can define $\tilde{f} : E^* \to \mathbb{C}$ (where $E^*$ is the punctured unit disk) by $\tilde{f}(q) := f(\tau)$. In particular, $q \to 0$ corresponds to $\tau \to \infty$.

2.4 DEFINITION. We say that $f$ is holomorphic at $\infty$ if $\tilde{f}$ is holomorphic at 0.
3. **Examples**

In other words, \( f \) is holomorphic at \( \infty \) if we can write \( \tilde{f} = \sum_{n \geq 0} a_n q^n \), or, equivalently, \( f(\tau) = \sum_{n \geq 0} a_n (e^{2\pi i \tau})^n \). This is how we make formal the request that \( f \) does not grow too fast for \( \tau \to \infty \).

2.5 Definition. Let \( k \in \mathbb{Z} \); a weakly modular form \( f \) of weight \( k \) is a modular form of weight \( k \) if \( f \) is holomorphic on \( \mathbb{H} \) and at \( \infty \). In this case, we define \( f(\infty) := a_0 \); \( f \) is called a cusp form if \( f(\infty) = 0 \).

Most interesting properties of modular forms are encoded in the Fourier coefficients \( a_n \).

2.6 Remark. Since \( -I \in \text{SL}(2, \mathbb{Z}) \), for a modular forms we have \( f(\tau) = f((-I)\tau) = (-1)^k f(\tau) \); in particular modular forms can exists only for \( k \) even.

3 **Examples**

3.1 Example (Eisenstein series). Let \( \Lambda \) be a lattice in \( \mathbb{C} \); then \( \sum_{\lambda \in \Lambda} 1/|\lambda|^\sigma \) is convergent for every \( \sigma > 2 \). Let \( k \geq 2 \); we define the Eisenstein series of weight \( k \) as

\[
G_k(\tau) := \frac{(k-1)!}{2(2\pi i)^k} \sum_{{m,n \in \mathbb{Z}}'} \frac{1}{(m\tau+n)^k},
\]

where the prime means that we exclude the value \((0,0)\). If \( k > 2 \) the series is absolutely convergent, in the case \( k = 2 \) we have to prove convergence with some other method. Assume \( k > 2 \); then we can rearrange the terms of the series and this allow us to prove that \( G_k \) is a modular form of weight \( k \) and holomorphic on \( \mathbb{H} \); the last thing to check is holomorphicity at \( \infty \). Thanks to Euler identity, for \( z \in \mathbb{H} \) we have

\[
\sum_{n \in \mathbb{Z}} \frac{1}{n+z} = \frac{\pi}{\tan(\pi z)} = -\pi i - 2\pi i \sum_{n \geq 1} \frac{e^{2\pi i n z}}{1 - e^{2\pi i n z}} = -\pi i - 2\pi i \sum_{n \geq 1} e^{2\pi i n z}.
\]
We can also consider (it is no more than computing a derivative)

\[
\sum_{n \in \mathbb{Z}} \frac{1}{(n+z)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i n z}.
\]

Substituting in the Eisenstein series we get

\[
G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \left( \sum_{n \in \mathbb{Z}} \frac{1}{n^k} + \sum_{m \in \mathbb{Z}} \sum \frac{1}{(m \tau + n)^k} \right) = \frac{(k-1)!}{(2\pi i)^k} \left( \sum_{n \geq 1} \frac{1}{n^k} + \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} \frac{1}{(m \tau + n)^k} \right) = \frac{(k-1)!}{(2\pi i)^k} \left( \sum_{n \geq 1} \frac{1}{n^k} + \sum_{m \geq 1} \frac{(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} n^{k-1} e^{2\pi i n m \tau} \right) = \frac{(k-1)!}{(2\pi i)^k} \zeta(k) + \sum_{m \geq 1} \left( \sum_{d | m} d^{k-1} \right) q^m = -\frac{B_k}{2k} + \sum_{n \geq 1} \sigma_{k-1}(n) q^n,
\]

where \(B_k\) is the \(k\)-th Bernoulli number and \(\sigma\) is the sum of divisor function; hence \(G_k\) is holomorphic at \(\infty\). In particular

\[
G_2(\tau) = -\frac{1}{24} + q + 3q^2 + \cdots,
\]

\[
G_4(\tau) = \frac{1}{246} + q + 9q^2 + \cdots.
\]

For \(k = 2\), the sum do not converge absolutely; we define

\[
G^*_k(\tau) := -\frac{1}{8\pi} \lim_{\varepsilon \to 0} \sum_{n,m \in \mathbb{Z}} \frac{1}{(m \tau + n + \varepsilon)^k} = G_k(\tau) + \frac{1}{8\pi \varepsilon^3}.
\]

The new series are absolutely convergent; but \(G^*_k\) is no more holomorphic since it depends explicitly on the imaginary part of \(\tau\). We can compute how the transformation property behaves on the correction term:

\[
G_k(A \tau) = (c \tau + d)^k G_k(\tau) - \frac{c(c \tau + d)}{4\pi i}.
\]

### 3.2 Example (Discriminant function)

We can define \(\Delta \colon \mathbb{H} \to \mathbb{C}\), the *discriminant function*, as \(\Delta(\tau) := q \prod_{n \geq 1} (1 - q^n)^{24}\), where as usual \(q = e^{2\pi i \tau}\). This converges on \(\mathbb{H}\); if it is modular, then it is a cusp form. Obviously \(\Delta(\tau + 1) = \Delta(\tau)\); define \(\Delta' := \frac{\delta}{\delta \tau}\) and let \(\Delta'/\Delta(\tau)\) the logarithmic derivative of \(\Delta\). We find
4. Zeros of modular forms

that

\[ \frac{\Delta'}{\Delta}(\tau) = 2\pi i \left(1 - 24 \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \right) = -24 \cdot 2\pi i G_k(\tau). \]

Then

\[ \frac{d}{d\tau} \log \left( \Delta \left( -\frac{1}{\tau} \right) \right) = \frac{1}{\tau^2} \frac{\Delta'}{\Delta}(\tau) + \frac{12}{\tau} = \frac{d}{d\tau} \log(\Delta(\tau)\tau^{12}), \]

that is \( \Delta(-\tau^{-1}) = \text{const} \cdot \tau^{12}\Delta(\tau) \). If we put \( \tau = i \), then \( -\tau^{-1} = \tau \) and \( \tau^{12} = 1 \), so the constant must be 1 and \( \Delta \) is a cusp form of weight 12.

3.3 Remark. We denote the vector space of modular forms of weight \( k \) with \( M_k \) and the vector space of cusp forms of weight \( k \) with \( S_k \). It is obvious that if \( f_k \in M_k \) and \( f_l \in M_l \) then \( f_k f_l \in M_{k+l} \).

4. Zeroes of modular forms

If \( f \) is a meromorphic function on \( \mathbb{H} \) we can define its order at a point \( p \in \mathbb{H} \) as \( v_p(f) \), the integer such that \( \frac{f(\tau)}{(\tau - p)^{v_p(f)}} \) is holomorphic and non-zero at \( p \). If \( f \) is a modular form, then \( f(\tau) = (c\tau + d)^{-k}f(A\tau) \) for every \( A \in \text{SL}(2,\mathbb{Z}) \), so \( v_p(f) = v_{Ap}(f) \) for every \( A \in \text{SL}(2,\mathbb{Z}) \). In particular, if \( f = \sum_{n \geq 0} a_n q^n \), then we define \( v_\infty(f) := v_0(f) \).

4.1 Theorem. Let \( f \) be a modular form of weight \( k \), then

(3) \[ v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_q(f) + \sum_{p \in \mathbb{H}/\Gamma \setminus \{\rho,i\}} v_p(f) = \frac{k}{12}. \]

In the stack interpretation, we define modular forms as sections of a line bundle \( \mathcal{L}_2 \to \mathcal{M}_{1,1} \cong \mathbb{P}(4,6) \); then the theorem says that the degree of this line bundle is \( k/12 \).

Proof. We can assume that our modular form \( f \) has no zeroes on the boundary of the fundamental domain \( D \) (except maybe in \( i \) or \( q \)), since we can move slightly \( D \) until this is true.

Now we can integrate \( df/f \) on the boundary of \( D \). More formally, consider Figure 2a: first, we integrate on a path like \( \gamma \) in such a way that all internal singularities are inside \( \gamma \); by the residue theorem,

\[ \frac{1}{2\pi i} \int_{\gamma} \frac{df}{f} = \sum_{p \in \mathbb{H}/\Gamma \setminus \{\rho,i\}} v_p(f). \]

We will compute now the same integrals piece by piece. For simplicity, we forget about the coefficient \( 2\pi i \).
• The integral on the arc near $q$ is just $-\frac{1}{6}v_q(f)$, since we can compute the integral along the path $\gamma_q$ of Figure 2b getting $-v_q(f)$ (since we are going clockwise this time) and then, passing to the limit of the radius, we have to divide by 6 since the angle is $\pi/3$.

• The same applies to the integral on the arc near $q^2$.

• With the same method, the integral on the arc near $i$ is $-\frac{1}{2}v_i(f)$.

• Using the transformation $\tau \mapsto q$, the horizontal segment becomes a whole clockwise circle around $q = 0$, so the integral on the segment is $-v_\infty(f)$.

• The two vertical path are obtained one from the other by applying $T$ or $T^{-1}$; since $f(T\tau) = f(\tau)$ and they are in opposite direction, the sum of the two integrals is 0.

• The two remaining arcs are obtained one from the other by applying $S$ or $S^{-1}$; this time, $f(S\tau) = \tau^k f(\tau)$, so

$$\frac{d f(S\tau)}{f(S\tau)} = k\frac{d \tau}{\tau} + \frac{d f(\tau)}{f(\tau)};$$

then, the sum of the two integral is

$$\int \left( \frac{d f(\tau)}{f(\tau)} - \frac{d f(S\tau)}{f(S\tau)} \right) = \int -k\frac{dz}{z} = -k\left( -\frac{1}{12} \right) = k\frac{1}{12}. $$

Comparing the two results we get

$$\sum_{p \in \mathbb{H}/\Gamma \setminus \{q\}} v_p(f) = -\frac{1}{3}v_q(f) - \frac{1}{2}v_i(f) - v_\infty(f) + k\frac{1}{12}. \quad \square$$

We recall that $M_k = 0$ for $k$ odd, that is, there are no odd weighted modular forms; moreover, since $G_{2k} \in M_{2k}$ is a modular forms that is not a cusp form.
4. **Zeroes of modular forms**

(Bernoulli numbers are always non-zero) it follows that \( \dim M_{2k}/S_{2k} \geq 1 \); but \( S_{2k} \) is the kernel of the map \( f \mapsto f(\infty) \), so \( \dim M_{2k}/S_{2k} \leq 1 \); hence, \( M_{2k} = S_{2k} \oplus \mathbb{C}G_{2k} \).

**4.2 Theorem.**

1. If \( k < 0 \) or \( k \) is odd, then \( M_k = 0 \).

2. For \( k \in \{0, 4, 6, 8, 10\} \), \( S_k = 0 \) and \( M_k = \mathbb{C}G_k \); \( G_0 = 1 \).

3. Multiplication by \( \Delta \) gives an isomorphism \( M_{k-12} \to S_k \) for all \( k \).

**Proof.** The first statement follows from equation (3), since all left-hand side terms are non-negative. We have \( M_2 = 0 \) since \( 1/6 \) cannot be written as a non-negative integral combination of \( 1/2 \) and \( 1/3 \); \( S_k = 0 \) for \( k < 12 \) is trivial since for a cusp form we have \( \nu_\infty(f) \geq 1 \).

Since \( \Delta \) has no zeroes on \( \mathbb{H} \), if \( f \in S_k \) we can write \( g := f/\Delta \) and \( g \) has weight \( k - 12 \). Now \( \nu_p(g) = \nu_p(f) \) for every \( p \in \mathbb{H} \) and \( \nu_\infty(g) = \nu_\infty(f) - 1 \), hence \( g \in M_{k-12} \). From this it follows the rest of the second statement.

**4.3 Corollary.** The dimension of \( M_k \) is

\[
\dim M_k = \begin{cases} 
0 & \text{if } k < 0 \text{ or } k \text{ odd;} \\
[k/12] & \text{if } k \equiv 2 \pmod{12}; \\
[k/12] + 1 & \text{if } k \not\equiv 2 \pmod{12}.
\end{cases}
\]

**4.4 Corollary.** Let \( M_A := \bigoplus_k M_k \); then as a graded ring \( M_A \cong \mathbb{C}[G_4, G_6] \). Equivalently, a basis of \( M_k \) is \( \{G_4^aG_6^b \mid 4a + 6b = k\} \).

**Proof.** In multiple steps.

- If \( k \leq 6 \) this is obvious.

- Since \( M_{12} = \mathbb{C}G_{12} \oplus \Delta \) and we have \( \lambda_4 G_4 + \lambda_6 G_6 \in M_{12} \) for every \( \lambda_4, \lambda_6 \in \mathbb{C} \), then the statement is true for \( M_{12} \) and in particular \( \Delta \) is generated by \( G_4 \) and \( G_6 \).

- By induction on even \( k \) greater than 6; choose \( a \) and \( b \) such that \( 4a + 6b = k \) and let \( g := G_4^aG_6^b \in M_k \); \( g \) is not a cusp form, so for every \( f \in M_k \) there exists \( \lambda \in \mathbb{C} \) such that \( f - \lambda g \) is a cusp form; but then \( f - \lambda g \in S_k = M_{k-12} \Delta \) and we conclude since both \( \Delta \) (by the previous point) and \( M_{k-12} \) (by induction) are generated by \( G_4 \) and \( G_6 \).

Define now \( E_k := G_k \cdot (-2k/8_k) = 1 + \cdots \).

**4.5 Corollary.**

\( E_4^2 = E_8 \).
By this corollary we can state the following non-trivial identity for every \( n > 0 \):

\[
\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).
\]

Another identity is \( E_4^3 - E_6^2 = 1728\Delta \).

## 5 Theta functions

Let \( \Lambda \) be a lattice in \( \mathbb{R}^n \), such that \( v \cdot v \in \mathbb{N} \) for every \( v \in \Lambda \). We wonder how many vectors of a given length exist in \( \Lambda \). We define a generating function

\[
\Theta_\Lambda(\tau) = \sum_{n \geq 0} \left| \{ v \in \Lambda \mid v \cdot v = n \} \right| q^n,
\]

where again \( q = e^{2\pi i \tau} \). We can write the same function in a simpler way:

\[
\Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{v \cdot v / 2}.
\]

We want to show that these are modular forms; to do this we make use of the Poisson summation formula.

Let \( \varphi : \mathbb{R}^n \to \mathbb{R} \) a smooth function rapidly decreasing at \( \infty \), that is, such that as \( \|x\| \to \infty \), it goes as \( \|x\|^{-c} \) for \( c \geq n \). The Fourier transform of \( \varphi \) is \( \hat{\varphi} : \mathbb{R}^n \to \mathbb{R} \) defined by

\[
\hat{\varphi}(t) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi i t \cdot x} \, dx.
\]

Let \( \mu \) the volume of \( \mathbb{R}^n / \Lambda \) (equivalent to \( \det(a_i \cdot a_j)^{n/2} \) where \( a_i \) is a basis of \( \Lambda \)); let \( \Lambda^\vee \) be the dual lattice, that is the set of all \( w \in \mathbb{R}^n \) such that \( w \cdot v \in \mathbb{Z} \) for every \( v \in \Lambda \).

### 5.1 Theorem (Poisson summation formula).

\[
\sum_{v \in \Lambda} \varphi(v) = \frac{1}{\mu} \sum_{w \in \Lambda^\vee} \hat{\varphi}(w).
\]

Let \( t \in \mathbb{R}_{>0} \) and define \( \tilde{\Theta}_\Lambda(t) := \sum_{v \in \Lambda} e^{-\pi tv \cdot v} \).

### 5.2 Proposition.

\[\tilde{\Theta}_\Lambda(t^{-1}) = t^{n/2} \mu \tilde{\Theta}_\Lambda(t).\]

**Proof.** Fix \( t \) and put \( f(x_1, \ldots, x_n) := e^{-\pi(x_1^2 + \cdots + x_n^2)} \). It is easy to prove that \( f \) is a rapidly decreasing function and that \( \hat{f} = f \). Consider the lattice \( \sqrt{t} \Lambda \); its dual is \( 1/\sqrt{t} \Lambda^\vee \) and its volume is \( t^{n/2} \mu \).

Applying the Poisson summation formula, we get

\[
\sum_{v \in \Lambda} e^{-\pi tv \cdot v} = \frac{t^{-n/2}}{\mu} \sum_{w \in \Lambda^\vee} e^{-\pi t^{1/2} w \cdot w}.
\]
This gives the statement. □

Assume from now on that \( \Lambda \) is a unimodular, even, integral lattice, that is, such that \( \Lambda' = \Lambda, v \cdot v \in 2\mathbb{Z} \) and \( w \cdot v \in \mathbb{Z} \) for every \( v, w \in \Lambda \).

5.3 Theorem.

1. \( \Theta_{\Lambda}(\tau) = \sum_{v \in \Lambda} q^{v \cdot \tau/2} \) is a modular form of weight \( n/2 \);
2. \( n \) is divisible by \( 8 \).

Proof. Since \( v \cdot v \in 2\mathbb{Z} \), the definition of \( \Theta_{\Lambda}(\tau) \) is a \( q \)-development; moreover it is clear that it is invariant under \( \tau \to \tau + 1 \). We want to prove that \( \Theta_{\Lambda}(-1/\tau) = (\tau)^{n/2} \Theta_{\Lambda}(\tau) \); this is enough because, if \( 8 \mid n \), the \( i \) go away and we remain with a modular form. Since \( \Theta_{\Lambda} \) is an analytic function, we can prove it just for \( \tau = it \) with \( t \in \mathbb{R}_{>0} \). Now, \( \Theta_{\Lambda}(it) = \sum_{v \in \Lambda} e^{-\tau v \cdot v} = \tilde{\Theta}_{\Lambda}(t) \); besides, \( \Theta_{\Lambda}(-1/\tau) = \tilde{\Theta}_{\Lambda}(-1/\tau) \). The statement then follows from Proposition 5.2.

Conversely, assume \( 8 \nmid n \); replacing \( \Lambda \) by \( \Lambda^2 \) or \( \Lambda^4 \) we may assume that \( n \equiv 4 \mod(8) \), so \( \Theta_{\Lambda}(-1/\tau) = -\tau^{n/2} \Theta_{\Lambda}(\tau) \). We recall that from every function \( f \) on \( \mathbb{H} \) we can define \( f|_kA(\tau) = (c\tau + d)^{-k}f(A\tau) \) for \( A \in \text{SL}(2, \mathbb{Z}) \). In particular, we apply this to \( f = \Theta_{\Lambda}, k = n/2 \) and \( A \in \{S, T\} \). We obtain respectively \( -\Theta_{\Lambda}(\tau) \) and \( \Theta_{\Lambda}(\tau) \); but \( (ST)^3 = I \), so

\[
\Theta_{\Lambda}(\tau) = \Theta_{\Lambda}|_{n/2, (ST)^3} = -\Theta_{\Lambda}(\tau),
\]

contradiction. □

5.4 Corollary. There is a cusp form \( f_{\Lambda} \) of weight \( n/2 \) such that \( \Theta_{\Lambda} = E_{n/2} + f_{\Lambda} \).

For \( n \equiv 0 \mod(8) \) it is quite easy to define a unimodular, even, integral lattice on \( \mathbb{R}^n \). For example, start with the lattice \( \Lambda_n := \{v \in \mathbb{Z}^n \mid v \cdot v \in 2\mathbb{Z}\} \) and consider \( \Lambda_n := \Lambda_n \oplus (1/2, \ldots, 1/2, \mathbb{Z}) \). This construction gives in particular \( \Lambda_8 = E_8 \).

5.5 Example. We have \( \Theta_{\Lambda_8} = E_4 \), since there is no cusp forms of weight 4. Besides, \( E_4 = 1 + 240 \sum_{n \geq 1} c_3(n)q^n \) and this gives us the number of lattice with the properties we wanted. In the same way, \( \Theta_{\Lambda_{16}} = \Theta_{\Lambda_8 \oplus \Lambda_8} = E_4^2 = E_8 \).

6 Modular forms for congruence subgroups

The group \( \text{SL}(2, \mathbb{Z}) \) contains copies of the integers: they are identified with the subgroups \( \Gamma(N) \) of matrices \( A \equiv I \mod(N) \); we have also the subgroups

\[
\Gamma^0(N) := \{A \in \text{SL}(2, \mathbb{Z}) \mid A \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \mod(N)\},
\]
\[
\Gamma_0(N) := \{A \in \text{SL}(2, \mathbb{Z}) \mid A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod(N)\}.
\]

6.1 Definition. A subgroup \( G \) of \( \text{SL}(2, \mathbb{Z}) \) is called a congruence subgroup if \( \Gamma(N) \subseteq G \)
6.2 Definition. Fixed a congruence subgroup $G$, a holomorphic function $f: \mathbb{H} \to \mathbb{C}$ is called a modular form of weight $k$ on $G$ if:

1. $f|_{k,A} = f$ for every $A \in G$ (that is, $f(\tau) = (c\tau + d)^{-k}f(A\tau)$);

2. $f$ is holomorphic at the cusps: for every $A \in \text{SL}(2,\mathbb{Z})$, there exists $l > 0$ such that $f|_{k,A} = \sum_{n\geq 0} a_n q^{n/l}$ with $a_n \in \mathbb{C}$ and $q^{n/l} = e^{2\pi i n\tau/l}$.

There is a geometric interpretation of the second condition.

- Let $\mathcal{Q} := \mathbb{Q} \cup \{\infty\}$; the action of $\text{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ extends to $\mathcal{Q}$ by $Aa = \frac{ax+b}{cx+d}$ (these action sends $\mathcal{Q}$ to itself). A cusp of $\mathbb{H}/G$ is an element of $\mathcal{Q}/G$; in particular, if $G = \text{SL}(2,\mathbb{Z})$ we have only one cusp which we can imagine to be $\infty$. In general, $\mathbb{H}/G$ can be compactified to a complete orbifold Riemann surface as $\mathbb{H}/G = \mathbb{H}/G \cup \{\text{cusps}\}$.

- Let $a \in \mathcal{Q}$ and $A \in \text{SL}(2,\mathbb{Z})$, with $A(\infty) = a$. Let $l \geq 0$ such that $T^l \in A^{-1}GA$; then
  \[ (f|_{k,A})|_{k,T^l} = f|_{k,AT^l} = f|_{k,A}, \]
  that is, $f|_{k,A}$ is mapped to itself by $\tau \to \tau + l$. We fix $l$ to be minimal with respect to his condition; this $l$ is called width of the cusp. Now we can write $f|_{k,A} = \sum_{n \in \mathbb{Z}} a_n q^{n/l}$, and holomorphic at cusp $a$ is equivalent to $a_n = 0$ for every $n < 0$.

- Geometrically, $\mathbb{H}/G$ is a complex orbifold that has an obvious map $\varphi$ to $\mathbb{H}/\text{SL}(2,\mathbb{Z})$; this map is a branch cover of degree $[\text{SL}(2,\mathbb{Z}) : G]$. The point $\infty$ in the target has as fiber the set of cusps in the source; moreover, the order of $\varphi$ at a cusp is just its width (that is, at $a$, $q^{n/l}$ is a local coordinate).

6.3 Example. Consider $G := \Gamma(2)/\{\pm 1\}$; it can be proved that it is the free group $\langle \{1, i\} \rangle$. A fundamental domain is represented in Figure 3. Its cusps then are 0, 1, $\infty$; the width of $\infty$ is 2. As before, we define the set of modular forms to be $M_{k,G}$ with the subspace $S_{k,G}$ of cusp forms (that is, modular forms such that $f(a) = 0$ for every cusp $a$). They are finite dimensional vector spaces and we can compute their dimensions.

6.4 Example. The theta function $\Theta_{2i}$ is $\sum_{n_1, n_2 \in \mathbb{Z}} q^{n_1^2 + n_2^2}$. This is not even, so it is not a modular form; but a similar argument of the one did before shows that it is a modular form on some subgroup, precisely a modular form of weight 2 on $\Gamma_0(4)$.

6.5 Corollary.

1. Every positive integer is the sum of four squares;

2. $|\{n_1, \ldots, n_4 \in \mathbb{Z} | \sum n_i^2 = n\}| = 8(\sum_{d|n, 4|d} d)$. 

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Proof. The first statement is obvious; for the second, consider $8G_2(\tau) - 32G_2(4\tau)$; this is a modular form of weight 2 on $\Gamma_0(4)$. This is quite surprising since $G_2$ is not even; but we recall that $G_2^*(\tau)$ is not holomorphic but transforms as a modular forms; so the one we are considering is just $8G_2^*(\tau) - 32G_2^*(4\tau)$.

### 7 Hecke theory

On modular forms there is an algebra of operators (the Hecke operators) such that there is a basis of simultaneous eigenvalues for the operators.

Recall that we have an isomorphism of vector spaces between:

- complex functions $F$ of oriented lattices $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\Im(\omega_1/\omega_2) > 0$ such that $F(a\Lambda) = a^{-k}\Lambda$;

- holomorphic functions $f: \mathbb{H} \to \mathbb{C}$ such that $f(A\tau) = (c\tau + d)^{-k}f(\tau)$ for every $A \in \text{SL}(2, \mathbb{Z})$.

In particular we associate to a morphism $F$ the function $f(\tau):= F(\tau\mathbb{Z} \oplus \mathbb{Z})$ and to a function $f$ the morphism such that $F(Z\omega_1 \oplus Z\omega_2) := \omega_2^{-k}f(\omega_1/\omega_2)$.

Let $F$ be a lattice function of weight $k$; define the operators $T_n := T_n^k$ by

$$T_n F(\Lambda) := n^{k-1} \sum_{\Lambda' \subseteq \Lambda | [\Lambda : \Lambda'] = n} F(\Lambda').$$

These $T_n$ have an interpretation as morphisms of moduli space of elliptic curves with additional level structure. Note anyway that $T_n F$ is a lattice function of weight $k$. Thus, denoting the corresponding function with $f: \mathbb{H} \to \mathbb{C}$, we define $T_n f(\tau) := T_n F(\tau\mathbb{Z} \oplus \mathbb{Z})$. Then for $T_n F$ to be a lattice function of weight $k$ means that $T_n f(A\tau) = (c\tau + d)^{-k}T_n f$. After some computation we obtain a description in terms of $\tau$: $T_n f(\tau) = n^{k-1} \sum_{A \in \mathcal{M}_n} (c\tau + d)^{-k}f(A\tau)$. The summation indices means that $A$ runs through a system of representatives of $\Gamma \setminus \mathcal{M}_n$, where $\mathcal{M}_n$ is the set of $2 \times 2$ matrices with entries in $\mathbb{Z}$ and determinant $n$, and SL$(2, \mathbb{Z})$ acts on $\mathcal{M}_n$ by multiplication on the left.

If $f \in M_k$, then $T_n f$ is holomorphic on $\mathbb{H}$, plus, we already seen that it transforms as a modular forms; to check that $T_n f$ is a modular form, we
need to prove that it is holomorphic at \( \infty \); we do this writing down its \( q \)-development.

7.1 Theorem.

1. Let \( f \in M_k \) with Fourier development \( f(\tau) = \sum_{n \geq 0} c(n) q^n \); then

\[
T_n f(\tau) = \sum_{m \geq 0} \left( \sum_{d \mid n} \sum_{d, m \geq 0} d^{k-1} \right) c(nm/d^2) q^m.
\]

In particular, \( T_n f \in M_k \) and if \( f \) is a cusp form, then \( T_n f \) is.

2. \( T_n \) satisfies

\[
T_m T_n = \sum_{d \mid n} \sum_{d, m \geq 0} d^{k-1} T_{nm/d};
\]

in particular, \( T_n \) and \( T_m \) commute and if \((m, n) = 1\), \( T_n T_m = T_{nm} \).

Proof. A system of representatives of \( \Gamma \backslash \mathcal{M}_n \) is the set of matrices \((a b ; 0 d)\) such that \( ad = n \) and \( 0 \leq b < d \). Then

\[
T_n f(\tau) = n^{k-1} \sum_{a, d > 0, ad = n} d^{d-1} \sum_{b = 0} d^{-k} f(\frac{ax+b}{d}).
\]

Substituting the \( q \)-development of \( f \) we obtain

\[
T_n f(\tau) = n^{k-1} \sum_{a, d > 0, ad = n} d^{d-1} \sum_{b = 0} d^{-k} \sum_{m \geq 0} c(m) e^{2\pi i \mu_{b/d}}.
\]

Note that

\[
\sum_{b = 0} d^{-1} e^{2\pi i mb/d} = \begin{cases} 0 & d \nmid m \\ d & d \mid m \end{cases}
\]

Now, observe that the second statement follows from the first by easy computations.

7.2 Remark. Observe that \( T_p T_p \circ T_p \circ \cdots = T_p^{n+1} + p^{k-1} T_p^{n-1} \) when \( p \) is prime; then if \( n = p_1^{n_1} \cdots p_l^{n_l} \), then \( T_n = T_{p_1^{n_1}} \cdots T_{p_l^{n_l}} \).

7.3 Definition. The Hecke operators are the operators \( T_n : M_k \to M_k \).

The Hecke operators are a set of commuting linear maps; it is possible then to search for common eigenvectors, that is, modular forms \( f \) such that \( T_n f = \lambda_n f \) for every \( n \geq 1 \).

7.4 Definition. Let \( f \in M_k \) be a common eigenvector of all \( T_n \); assume \( f(\tau) = \sum_{n \geq 0} a_n q^n \) with \( a_n = 1 \); then \( f \) is called a Hecke form.

The condition on \( a_n = 1 \) is to normalize the form. At first it appears that the parameter \( q \) is not so special; but it happens that the coefficients of the Fourier transform have a geometrical meaning; in particular, they are related to the eigenvalues.
7.5 corollary. Let $f = \sum_{n \geq 0} a_n q^n$ be a Hecke form; then $T_n f = a_n f$ for all $n \geq 1$.

7.6 corollary. Let $f = \sum_{n \geq 0} a_n q^n$ be a Hecke form; then $a_n a_m = \sum_{d | n, d | m} a_{d/m} d^{k-1} c(nm/d^2)$.

In particular, if $(n, m) = 1$, then $a_n a_m = a_{nm}$.

Proof. It is obvious by the formula of $T_n f(\tau)$. Since $T_n f(\tau) = \lambda_n f(\tau)$ and $a_1 = 1$, we can write

$$\lambda_n = \lambda_n a_1 = \sum_{d | n, d | 1} d^{k-1} c(nm/d^2) = a_n.$$ 

7.7 example. The Eisenstein series $G_k$ is a Hecke form for $k \geq 4$; so it is $\Delta$.

7.8 corollary.

$$\tau(n) \lambda(n) = \sum_{d | n, d | m} d^{11} c(nm/d^2).$$

7.9 theorem. The Hecke forms form a basis for $M_k$ for all $k$.

8 L-series

Let $f = \sum_{n \geq 0} a_n q^n$ be a Hecke form; we can associate to it its L-series

$$L(f, s) := \sum_{n \geq 1} \frac{a_n}{n^s}$$

that converges absolutely and uniformly for $\Re s > k$ and is a holomorphic function for $\Re s > 0$. To prove these convergency results we need some machinery.

8.1 lemma. Let $f = \sum_{n \geq 0} a_n q^n \in M_k$; then $a_n \in O(n^{k-1})$ (that is, $a_n/n^{k-1} \to 0$ as $n \to \infty$).

8.2 corollary. We can write $L(f, s)$ as an Euler product:

$$L(f, s) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$ 

Proof. From $a_n a_m = a_{nm}$ if $(n, m) = 1$, it follows that $a_{p_{1}^{n_{1}} \cdots p_{l}^{n_{l}}} = a_{p_{1}^{n_{1}}} \cdots a_{p_{l}^{n_{l}}}$; then $L(f, s) = \prod_{p \text{ prime}} \sum_{n \geq 0} a_{p^n} p^{-ns}$. We have to show that $\sum_{n \geq 0} a_{p^n} p^{-ns} = (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$. We know that $a_{p^{m+1}} - a_p a_{p^n} + p^{k-1} a_{p^{m-1}} = 0$ for $p$ prime; if we multiply the series with $1 - a_p p^{-s} + p^{k-1-2s}$, we see that the constant term, with respect to $t := p^{-s}$ is 1, that the second term is 0 and by induction we get that all other coefficients are 0 using the previous relation.

8.3 example. We can write the Riemann Zeta function $\zeta(s) = \sum_{n \geq 1} 1/n^s$ as $\prod_{p \text{ prime}} 1/(1 - p^{-s})$. We can compute $L(G_k, s)$; if $p$ is a prime, $a_{k-1}(p) = 1 + p^{k-1}$.
and the denominator of the terms of the L-series is \(1 + \sigma_{k-1}(p)p^{-s} + p^{k-1-2s} = (1 - p^{k-1-s})(1 - p^{-s})\) and it follows that

\[
L(G_k, s) = \prod_{p \text{ primes}} \left(\frac{1}{1-p^s}\right) \left(\frac{1}{1-p^{k-1-s}}\right) = \zeta(s)\zeta(s-k+1).
\]

8.4 Theorem. If \(f\) is a Hecke form of weight \(k\), then \(L(f, s)\) has a meromorphic continuation to the whole \(\mathbb{C}\) and satisfies some functional equation; if \(f\) is a cusp form, then \(L(f, s)\) is an entire function; otherwise it has a simple pole at \(s = k\).

References