Introduction to modular forms
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Contents

1 Introduction

Lecture 1 (2 hours) March 9th, 2009

Let E_Λ be the elliptic curve associated to lattice $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, oriented in the sense that $\Im \omega_1/\omega_2 > 0$. We know that $E_{\Lambda_1} \cong E_{\Lambda_2}$ if and only if $\Lambda_1 = a\Lambda_2$ for some $a \in \mathbb{C} \setminus \{0\}$.

1.1 definition. A *modular function* is a function $M_{1,1} \to \mathbb{C}$ where $M_{1,1}$ is the space of elliptic curves over $\mathbb{C}$.

A modular function can be viewed as a function

$$F\colon \{\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2\} \to \mathbb{C}$$

with the property that $F(a\Lambda) = F(\Lambda)$ for all lattice Λ and $a \in \mathbb{C} \setminus \{0\}$.

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1.2 definition. A *modular form of weight k*, with $k \in \mathbb{Z}$, is a modular function F such that

(1) $\quad F(a\Lambda) = a^{-k}F(\Lambda).$

Let $\mathrm{SL}(2,\mathbb{Z})$ the set of integral matrices with determinant 1; from now on, we will denote an element of $\mathrm{SL}(2,\mathbb{Z})$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $\Lambda := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ a lattice; if $A \in \mathrm{SL}(2,\mathbb{Z})$, $A(\omega_1, \omega_2)$ is just another basis for the same lattice. Let $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im\tau > 0\}$ be the half complex plane with positive imaginary part. The matrix group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ by $A(\tau) := \frac{a\tau+b}{c\tau+d}$.

Now let F be a modular form of weight k; we can associate to it a function $f\colon \mathbb{H} \to \mathbb{C}$ by $f(\tau) := F(\mathbb{Z}\tau \oplus \mathbb{Z})$; in this context, condition (1) ensure that

$$\begin{aligned} f(A(\tau)) &= F\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d} \oplus \mathbb{Z}\right) = F\left(\frac{1}{c\tau+d}(\mathbb{Z}(a\tau+b) \oplus \mathbb{Z}(c\tau+d))\right) = \\ &= (c\tau+d)^k F(A(\mathbb{Z}\tau \oplus \mathbb{Z})) = (c\tau+d)^k F(\mathbb{Z}\tau \oplus \mathbb{Z}) = \\ &= (c\tau+d)^k f(\tau). \end{aligned} \tag{2}$$

Conversely, if we have $f\colon \mathbb{H} \to \mathbb{C}$ such that $f(A(\tau)) = (c\tau+d)^k f(\tau)$, then we can associate to it a function F as in $F(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) = \omega_2^{-k} f(\omega_1/\omega_2)$; this is a modular form. In particular one obtain that these correspondences are each the inverse of the other. So we can give the following equivalent definition.

1.3 definition. A *modular form of weight k*, with $k \in \mathbb{Z}$, is a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ satisfying (2), and not growing too fast as $\tau \to \infty$.

The last condition will ensure later that modular forms corresponds to sections of a line bundle on $\overline{M}_{1,1}$. Another way to say the same thing is to define for every $f\colon \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}$, and $A \in \mathrm{SL}(2,\mathbb{Z})$ the function $f|_{k,A}\colon \mathbb{H} \to \mathbb{C}$ with $f|_{k,A}(\tau) := (c\tau+d)^{-k} f(A\tau)$; then we request $f = f|_{k,A}$ for every A.

Why modular forms are useful in mathematics?

1. There are very few modular forms; the space of modular forms of weight k is a vector space of finite dimension.
2. They occur naturally in many fields of mathematics and physics.

2 The modular group

Consider the previously defined action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$; since $-I$ acts trivially, we can also say that $\Gamma := \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$ acts on $\mathbb{H}$. We define two special elements:

1. $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, such that $S\tau = -\tau^{-1}$;
2. $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, such that $T\tau = \tau + 1$.

Moreover, we have $S^2 = I = (ST)^3$ in Γ.

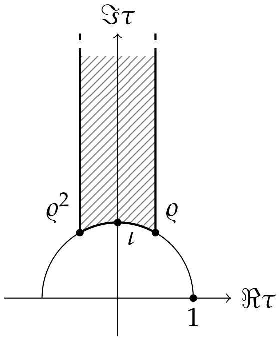


Figure 1: A fundamental domain of the action of Γ on $\mathbb{H}$.

Now we can find a fundamental domain for the action of Γ:

$$D := \{\tau \in \mathbb{H} \mid |\tau| \geq 1, -1/2 \leq \Re\tau \leq 1/2\}.$$

We define $\varrho := e^{2\pi\iota/3}$, so that $-\overline{\varrho} = \varrho^2 = e^{4\pi\iota/3}$. Pictorially, we have Figure 1

2.1 PROPOSITION. *The so defined D is a fundamental domain for the action of Γ on $\mathbb{H}$; in particular we have that:*

1. *every point in $\mathbb{H}$ has a conjugate point in D with respect to the action;*
2. *if $\tau, \tau' \in D$ are conjugate and different, then $\Re\tau = \pm 1/2$ and $\tau' = \tau \pm 1$, or $|\tau| = 1$, $\tau' = -1/\tau$;*
3. *let $\tau \in D$ and $I(\tau) := |\{A \in \Gamma \mid A\tau = \tau\}|$; then $I(\tau) = 1$ unless $\tau = \iota$ ($I(\iota) = 2$) or $\tau \in \{\varrho, -\overline{\varrho}\}$ ($I(\varrho) = I(-\overline{\varrho}) = 3$).*

We can translate this situation into the stack language; the closure of $\mathbb{H}/\Gamma$ can be identified with the weighted projective space $\mathbb{P}(4,6)$; so we can define a generic $1/2 : 1$ map to $\mathbb{P}^1$ (with two special point, corresponding to ι and ϱ).

2.2 DEFINITION. Let $k \in \mathbb{Z}$; a *weakly modular form of weight k* is a meromorphic function $f\colon \mathbb{H} \to \mathbb{C}$ such that $f(A\tau) = (c\tau + d)^k f(\tau)$ for every $A \in \mathrm{SL}(2,\mathbb{Z})$.

Let L be the space of meromorphic function $f\colon \mathbb{H} \to \mathbb{C}$; $\mathrm{SL}(2,\mathbb{Z})$ acts on L in this way: *TODO*

2.3 COROLLARY. *A function f is weakly modular of weight k if and only if $f(-\tau^{-1}) = f(\tau)$ and $f(\tau+1) = f(\tau)$ (e.g. if it is invariant with respect to S and T).*

Given τ, write $q := e^{2\pi\iota\tau}$; consider a weakly modular form of weight k; then we can define $\widetilde{f}\colon E^\star \to \mathbb{C}$ (where $E^\star$ is the punctured unit disk) by $\widetilde{f}(q) := f(\tau)$. In particular, $q \to 0$ corresponds to $\tau \to \infty$.

2.4 DEFINITION. We say that f is *holomorphic at ∞* if $\widetilde{f}$ is holomorphic at 0.

In other words, f is holomorphic at ∞ if we can write $\widetilde{f} = \sum_{n\geq 0} a_n q^n$, or, equivalently, $f(\tau) = \sum_{n\geq 0} a_n (e^{2\pi\imath\tau})^n$. This is how we make formal the request that f does not grow too fast for $\tau \to \infty$.

2.5 definition. Let $k \in \mathbb{Z}$; a weakly modular form f of weight k is a *modular form of weight* k if f is holomorphic on $\mathbb{H}$ and at ∞. In this case, we define $f(\infty) := a_0$; f is called a *cusp form* if $f(\infty) = 0$.

Most interesting properties of modular forms are encoded in the Fourier coefficients a_n.

2.6 remark. Since $-I \in \mathrm{SL}(2,\mathbb{Z})$, for a modular forms we have $f(\tau) = f((-I)\tau) = (-1)^k f(\tau)$; in particular modular forms can exists only for k even.

3 Examples

3.1 example (Eisenstein series). Let Λ be a lattice in $\mathbb{C}$; then $\sum_{\lambda\in\Lambda} 1/|\lambda|^\sigma$ is convergent for every $\sigma > 2$. Let $k \geq 2$; we define the Eistenstein series of weight k as

$$G_k(\tau) := \frac{(k-1)!}{2(2\pi\imath)^k} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(m\tau+n)^k},$$

where the prime means that we exclude the value $(0,0)$. If $k > 2$ the series is absolutely convergent, in the case $k = 2$ we have to prove convergence with some other method. Assume $k > 2$; then we can rearrange the terms of the series and this allow us to prove that G_k is a modular form of weight k. Indeed,

$$(c\tau + d)^k G_k(A\tau) = G_k(\tau)$$

since, disregarding multiplicative coefficients, we have

$$\begin{aligned}(c\tau+d)^k \sum_{m,n\in\mathbb{Z}} \frac{1}{(m(A\tau)+n)^k} &= \sum_{n,m\in\mathbb{Z}} \frac{1}{(m(a\tau+b)+n(c\tau+d))^k} = \\ &= \sum_{n,m\in\mathbb{Z}} \frac{1}{(m\tau+n)^k}\end{aligned}$$

since $(a\tau+b, c\tau+d)$ is another basis of the lattice $\mathbb{Z}\tau \oplus \mathbb{Z}$. This tell us that G_k is a weakly modular form of weight k and holomorphic on $\mathbb{H}$; the last thing to check is holomorphicity at ∞. Thanks to Euler identity, for $z \in \mathbb{H}$ we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+z} = \frac{\pi}{\tan(\pi z)} = -\pi\imath - 2\pi\imath\frac{e^{2\pi\imath z}}{1-e^{2\pi\imath z}} = -\pi\imath - 2\pi\imath \sum_{n\geq 1} e^{2\pi\imath n z}.$$

We can also consider (it is no more than computing a derivative)

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n+z)^k} = \frac{(-2\pi\iota)^k}{(k-1)!} \sum_{n\geq 1} n^{k-1} e^{2\pi\iota n z}.$$

Substituting in the Eisenstein series we get

$$\begin{aligned}
G_k(\tau) &= \frac{(k-1)!}{2(2\pi\iota)^k} \left(\sum_{n\in\mathbb{Z}} \frac{1}{n^k} + \sum_{m\in\mathbb{Z}}{}' \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} \right) = \\
&= \frac{(k-1)!}{(2\pi\iota)^k} \left(\sum_{n\geq 1} \frac{1}{n^k} + \sum_{m\geq 1} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^k} \right) = \\
&= \frac{(k-1)!}{(2\pi\iota)^k} \left(\sum_{n\geq 1} \frac{1}{n^k} + \sum_{m\geq 1} \frac{(-2\pi\iota)^k}{(k-1)!} \sum_{n\geq 1} n^{k-1} e^{2\pi\iota n m \tau} \right) = \\
&= \frac{(k-1)!}{(2\pi\iota)^k} \zeta(k) + \sum_{m\geq 1} \left(\sum_{d|m} d^{k-1} \right) q^m = \\
&= -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n,
\end{aligned}$$

where B_k is the k-th Bernoulli number and σ is the sum of divisor function; hence G_k is holomorphic at ∞. In particular *TODO*

$$\begin{aligned}
G_2(\tau) &= -\frac{1}{24} + q + 3q^2 + \cdots, \\
G_4(\tau) &= \frac{1}{246} + q + 9q^2 + \cdots.
\end{aligned}$$

For $k = 2$, the sum do not converge absolutely; we define

$$G_k^\star(\tau) := -\frac{1}{8\pi} \lim_{\varepsilon\to 0} \sum_{n,m\in\mathbb{Z}} \frac{1}{(m\tau+n)^k |m\tau+n|^\varepsilon} = G_k(\tau) + \frac{1}{8\pi\Im\tau}.$$

The new series are absolutely convergent; but $G_k^\star$ is no more holomorphic since it depends explicitly on the imaginary part of τ. We can compute how the transformation property behaves on the correction term:

$$G_k(A\tau) = (c\tau+d)^k G_k(\tau) - \frac{c(c\tau+d)}{4\pi\iota}.$$

3.2 EXAMPLE (Discriminant function). We can define $\Delta\colon \mathbb{H} \to \mathbb{C}$, the *discriminant function*, as $\Delta(\tau) := q\prod_{j\geq 1}(1-q^n)^{24}$, where as usual $q = e^{2\pi\iota\tau}$. This converges on $\mathbb{H}$; if it is modular, then it is a cusp form. Obviously $\Delta(\tau+1) = \Delta(\tau)$; define $\Delta' := \frac{\partial\Delta}{\partial\tau}$ and let ${}^{\Delta'}/_{\Delta}(\tau)$ the logarithmic derivative of Δ. We find that

$$\frac{\Delta'}{\Delta}(\tau) = 2\pi\iota\left(1 - 24\underbrace{\sum_{n\geq 1}\frac{nq^n}{1-q^n}}_{\sum_{n\geq 1}\sum_{i\geq 1} iq^{ni} = \sum_{m\geq 1}\sigma_k(m)q^m}\right) = -24\cdot 2\pi\iota G_k(\tau).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}\,\tau}\log\left(\Delta\left(-\frac{1}{\tau}\right)\right) = \frac{1}{\tau^2}\frac{\Delta'}{\Delta}\left(-\frac{1}{\tau}\right) = \frac{\Delta'}{\Delta}(\tau) + \frac{12}{\tau} = \frac{\mathrm{d}}{\mathrm{d}\,\tau}\log(\Delta(\tau)\tau^{12}),$$

that is $\Delta(-\tau^{-1}) = \mathrm{const}\cdot\tau^{12}\Delta(\tau)$. If we put $\tau = \iota$, then $-\tau^{-1} = \tau$ and $\tau^{12} = 1$, so the constant must be 1 and Δ is a cusp form of weight 12.

Lecture 2 (2 hours) March 12th, 2009

3.3 remark. We denote the vector space of modular forms of weight k with M_k and the vecor space of cusp forms of weight k with S_k. It is obvious that if $f_k \in M_k$ and $f_l \in M_l$ then $f_k f_l \in M_{k+l}$.

4 Zeroes of modular forms

If f is a meromorphic function on $\mathbb{H}$ we can define its order at a point $p \in \mathbb{H}$ as $v_p(f)$, the integer such that $f(\tau)(\tau - p)^{-v_p(f)}$ is holomorphic and non-zero at p. If f is a modular form, then $f(\tau) = (c\tau + d)^{-k} f(A\tau)$ for every $A \in \mathrm{SL}(2,\mathbb{Z})$, so $v_p(f) = v_{Ap}(f)$ for every $A \in \mathrm{SL}(2,\mathbb{Z})$. In particular, if $f = \sum_{n\geq 0} a_n q^n$, then we define $v_\infty(f) := v_0(\widetilde{f})$.

4.1 theorem. *Let f be a modular form of weight k, then*

$$v_\infty(f) + \frac{1}{2}v_\iota(f) + \frac{1}{3}v_\varrho(f) + \sum_{p\in\mathbb{H}/\Gamma\setminus\{\varrho,i\}} v_p(f) = \frac{k}{12}. \tag{3}$$

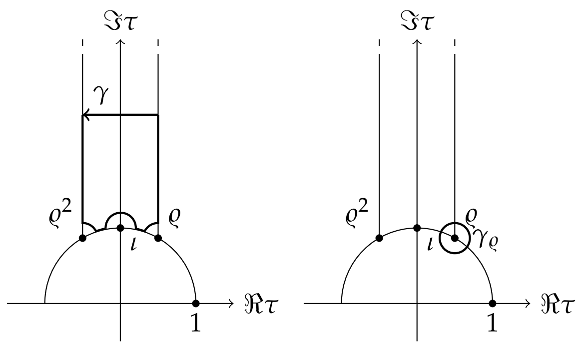
In the stack interpretation, we define modular forms as sections of a line bundle $\mathscr{L}_2 \to \overline{M}_{1,1} \cong \mathbb{P}(4,6)$; then the theorem says that the degree of this line bundle is $k/12$.

Proof. We can assume that our modular form f has no zeroes on the boundary of the fundamental domain D (except maybe in ι or ϱ), since we can move slightly D until this is true.

Now we can integrate $\mathrm{d}f/f$ on the boundary of D. More formally, consider Figure 2a: first, we integrate on a path like γ in such a way that all internal singularities are inside γ; by the residue theorem,

$$\frac{1}{2\pi\iota}\int_\gamma \frac{\mathrm{d}\,f}{f} = \sum_{p\in\mathbb{H}/\Gamma\setminus\{\varrho,i\}} v_p(f).$$

We will compute now the same integrals piece by piece. For simplicity, we forget about the coefficient $2\pi\iota$.



(a) Points in the inside. (b) The point ϱ.

Figure 2: Proof of Theorem 4.1.

- The integral on the arc near ϱ is just $-1/6 v_\varrho(f)$, since we can compute the integral along the path γ_ϱ of Figure 2b getting $-v_\varrho(f)$ (since we are going clockwise this time) and then, passing to the limit of the radius, we have to divide by 6 since the angle is $\pi/3$.
- The same applies to the integral on the arc near ϱ^2.
- With the same method, the integral on the arc near ι is $-1/2 v_\iota(f)$.
- Using the transformation $\tau \mapsto q$, the horizontal segment becomes a whole clockwise circle around $q = 0$, so the integral on the segment is $-v_\infty(f)$.
- The two vertical path are obtained one from the other by applying T or T^{-1}; since $f(T\tau) = f(\tau)$ and they are in opposite direction, the sum of the two integrals is 0.
- The two remaining arcs are obtained one from the other by applying S or S^{-1}; this time, $f(S\tau) = \tau^k f(\tau)$, so

  $$\frac{\mathrm{d}\, f(S\tau)}{f(S\tau)} = k\frac{\mathrm{d}\,\tau}{\tau} + \frac{\mathrm{d}\, f(\tau)}{f(\tau)};$$

  then, the sum of the two integral is

  $$\int \Big(\frac{\mathrm{d}\, f(\tau)}{f(\tau)} - \frac{\mathrm{d}\, f(S\tau)}{f(S\tau)}\Big) = \int -k\frac{\mathrm{d}\, z}{z} = -k\left(-\frac{1}{12}\right) = \frac{k}{12}.$$

Comparing the two results we get

$$\sum_{p \in \mathbb{H}/\Gamma \setminus \{\varrho, \iota\}} v_p(f) = -\frac{1}{3} v_\varrho(f) - \frac{1}{2} v_\iota(f) - v_\infty(f) + \frac{k}{12}. \qquad \square$$

We recall that $M_k = 0$ for k odd, that is, there are no odd weighted modular forms; moreover, since $G_{2k} \in M_{2k}$ is a modular forms that is not a cusp form

(Bernoulli numbers are always non-zero) it follows that $\dim M_{2k}/S_{2k} \geq 1$; but S_{2k} is the kernel of the map $f \mapsto f(\infty)$, so $\dim M_{2k}/S_{2k} \leq 1$; hence, $M_{2k} = S_{2k} \oplus \mathbb{C}G_{2k}$.

4.2 theorem.

1. *If $k < 0$ or k is odd, then $M_k = 0$.*

2. *For $k \in \{0, 4, 6, 8, 10\}$, $S_k = 0$ and $M_k = \mathbb{C}G_k$; $M_2 = 0$; $G_0 = 1$.*

3. *Multiplication by Δ gives an isomorphism $M_{k-12} \to S_k$ for all k.*

Proof. The first statement follows from equation (3), since all left-hand side terms are non-negative. We have $M_2 = 0$ since $1/6$ cannot be written as a non-negative integral combination of 1, $1/2$ and $1/3$; $S_k = 0$ for $k < 12$ is trivial since for a cusp form we have $v_\infty(f) \geq 1$.

Since Δ has no zeroes on $\mathbb{H}$, if $f \in S_k$ we can write $g := f/\Delta$ and g has weight $k - 12$. Now $v_p(g) = v_p(f)$ for every $p \in \mathbb{H}$ and $v_\infty(g) = v_\infty(f) - 1$, hence $g \in M_{k-12}$. From this it follows the rest of the second statement. $\square$

4.3 corollary. *The dimension of M_k is*

$$\dim M_k = \begin{cases} 0 & \text{if } k < 0 \text{ or } k \text{ odd;} \\ \lfloor k/12 \rfloor & \text{if } k \equiv 2 \quad (12); \\ \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 \quad (12). \end{cases}$$

4.4 corollary. *Let $M_A := \bigoplus_k M_k$; then as a graded ring $M_A \cong \mathbb{C}[G_4, G_6]$. Equivalently, a basis of M_k is $\{G_4^a G_6^b \mid 4a + 6b = k\}$.*

Proof. In multiple steps.

- If $k \leq 6$ this is obvious.

- Since $M_{12} = \mathbb{C}G_{12} \oplus \Delta$, and we have $\lambda_4 G_4 + \lambda_6 G_6 \in M_{12}$ for every $\lambda_4, \lambda_6 \in \mathbb{C}$, then the statement is true for M_{12} and in particular Δ is generated by G_4 and G_6.

- By induction on even k greater than 6: choose a and b such that $4a + 6b = k$ and let $g := G_4^a G_6^b \in M_k$; g is not a cusp form, so for every $f \in M_k$ there exists $\lambda \in \mathbb{C}$ such that $f - \lambda g$ is a cusp form; but then $f - \lambda g \in S_k = M_{k-12}\Delta$ and we conclude since both Δ (by the previous point) and M_{k-12} (by induction) are generated by G_4 and G_6. $\square$

Define now $E_k := G_k \cdot (-2k/B_k) = 1 + \cdots$.

4.5 corollary.

$E_4^2 = E_8$.

By this corollary we can state the following non-trivial identity for every $n > 0$:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

Another identity is $E_4^3 - E_6^2 = 1728\Delta$.

5 Theta functions

Let Λ be a lattice in $\mathbb{R}^n$, such that $v \cdot v \in \mathbb{N}$ for every $v \in \Lambda$. We wonder how many vectors of a given length exist in Λ. We define a generating function

$$\Theta_\Lambda(\tau) = \sum_{n \geq 0} |\{v \in \Lambda \mid v \cdot v = n\}| \, q^{n/2},$$

where again $q = e^{2\pi\imath\tau}$. We can write the same function in a simpler way: $\Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{v \cdot v/2}$. We want to show that these are modular forms; to do this we make use of the Poisson summation formula.

Let $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ a smooth function rapidly decreasing at ∞, that is, such that as $\|x\| \to \infty$, it goes as $\|x\|^{-c}$ for $c \geq n$. The Fourier transform of φ is $\hat{\varphi} \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{\varphi}(t) = \int_{\mathbb{R}^n} \varphi(x) e^{-2\pi\imath t x} \, \mathrm{d}\, x.$$

Let μ the volume of $\mathbb{R}^n/\Lambda$ (equivalent to $\det(a_i \cdot a_j)^{n/2}$ where a_i is a basis of Λ); let $\Lambda^\vee$ be the dual lattice, that is the set of all $w \in \mathbb{R}^n$ such that $w \cdot v \in \mathbb{Z}$ for every $v \in \Lambda$.

5.1 theorem (Poisson summation formula).

$$\sum_{v \in \Lambda} \varphi(v) = \frac{1}{\mu} \sum_{w \in \Lambda^\vee} \hat{\varphi}(w).$$

Let $t \in \mathbb{R}_{>0}$ and define $\widetilde{\Theta}_\Lambda(t) := \sum_{v \in \Lambda} e^{-\pi t v \cdot v}$.

5.2 proposition.

$$\widetilde{\Theta}_{\Lambda^\vee}(t^{-1}) = t^{n/2} \mu \widetilde{\Theta}_\Lambda(t).$$

Proof. Fix t and put $f(x_1, \ldots, x_n) := e^{-\pi(x_1^2 + \cdots + x_n^2)}$. It is easy to prove that f is a rapidly decreasing function and that $\widetilde{f} = f$. Consider the lattice $\sqrt{t}\Lambda$; its dual is $1/\sqrt{t}\Lambda^\vee$ and its volume is $t^{n/2}\mu$.

Applying the Poisson summation formula, we get

$$\sum_{v \in \Lambda} e^{-\pi t v \cdot v} = \frac{t^{-n/2}}{\mu} \sum_{w \in \Lambda^\vee} e^{-\pi 1/t w \cdot w}.$$

This gives the statement. □

Assume from now on that Λ is a unimodular, even, integral lattice, that is, such that $\Lambda^\vee = \Lambda$, $v \cdot v \in 2\mathbb{Z}$ and $w \cdot v \in \mathbb{Z}$ for every $v, w \in \Lambda$.

5.3 theorem.

1. *$\Theta_\Lambda(\tau) = \sum_{v \in \Lambda} q^{v\cdot v/2}$ is a modular form of weight $n/2$;*

2. *n is divisible by 8.*

Proof. Since $v \cdot v \in 2\mathbb{Z}$, the definition of $\Theta_\Lambda(\tau)$ is a q-development; moreover it is clear that it is invariant under $\tau \to \tau + 1$. We want to prove that $\Theta_\Lambda(-1/\tau) = (\imath\tau)^{n/2}\Theta_\Lambda(\tau)$; this is enough because, if $8 \mid n$, the $\imath$ go away and we remain with a modular form. Since Θ_Λ is an analytic function, we can prove it just for $\tau = \imath t$ with $t \in \mathbb{R}_{>0}$. Now, $\Theta_\Lambda(\imath t) = \sum_{v\in\Lambda} e^{-\pi t v\cdot v} = \widetilde{\Theta}_\Lambda(t)$; besides, $\Theta_\Lambda(-1/\imath\tau) = \widetilde{\Theta}_\Lambda(-1/t)$. The statement then follows from Proposition 5.2.

Conversely, assume $8 \nmid n$; replacing Λ by Λ^2 or Λ^4 we may assume that $n \equiv 4 \pmod 8$, so $\Theta_\Lambda(-1/\tau) = -\tau^{n/2}\Theta_\Lambda(\tau)$. We recall that from every function f on $\mathbb{H}$ we can define $f|_{k,A}(\tau) = (c\tau+d)^{-k} f(A\tau)$ for $A \in \mathrm{SL}(2,\mathbb{Z})$. In particular, we apply this to $f = \Theta_\Lambda$, $k = n/2$ and $A \in \{S, T\}$. We obtain respectively $-\Theta_\Lambda(\tau)$ and $\Theta_\Lambda(\tau)$; but $(ST)^3 = I$, so

$$\Theta_\Lambda(\tau) = \Theta_\Lambda|_{n/2,(ST)^3} = -\Theta_\Lambda(\tau),$$

TODO

contraddiction. □

5.4 corollary. *There is a cusp form f_Λ of weight $n/2$ such that $\Theta_\Lambda = E_{n/2} + f_\Lambda$.*

For $n \equiv 0 \pmod 8$ it is quite easy to define a unimodular, even, integral lattice on $\mathbb{R}^n$. For example, start with the lattice $B_n := \{v \in \mathbb{Z}^n \mid v \cdot v \in 2\mathbb{Z}\}$ and consider $\Lambda_n := B_n \oplus (1/2, \ldots, 1/2)\mathbb{Z}$. This construction gives in particular $\Lambda_8 = E_8$.

5.5 example. We have $\Theta_{\Lambda_8} = E_4$, since there is no cusp forms of weight 4. Besides, $E_4 = 1 + 240\sum_{n\geq 1}\sigma_3(n)q^n$ and this gives us the number of lattice with the properties we wanted. In the same way, $\Theta_{\Lambda_{16}} = \Theta_{\Lambda_8\oplus\Lambda_8} = E_4^2 = E_8$.

6 Modular forms for congruence subgroups

Lecture 3 (2 hours)
March 16^{th}, 2009

The group $\mathrm{SL}(2,\mathbb{Z})$ contains copies of the integers: they are identified with the subgroups $\Gamma(N)$ of matrices $A \equiv I \pmod N$; we have also the subgroups

$$\Gamma^0(N) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} \star & 0 \\ \star & \star \end{smallmatrix}\right) \pmod N\},$$
$$\Gamma_0(N) := \{A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv \left(\begin{smallmatrix} \star & \star \\ 0 & \star \end{smallmatrix}\right) \pmod N\}.$$

6.1 definition. A subgroup G of $\mathrm{SL}(2,\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma(N) \subseteq G$

6.2 DEFINITION. Fixed a congruenge subgroup G, a holomorphic function $f\colon \mathbb{H} \to \mathbb{C}$ is called a *modular form of weight k on G* if:

1. $f|_{k,A} = f$ for every $A \in G$ (that is, $f(\tau) = (c\tau + d)^{-k} f(A\tau)$);

2. f is holomorphic at the cusps: for every $A \in \mathrm{SL}(2,\mathbb{Z})$, there exists $l > 0$ such that $f|_{k,A} = \sum_{n\geq 0} a_n q^{n/l}$ with $a_n \in \mathbb{C}$ and $q^{n/l} = e^{2\pi\iota\tau n/l}$.

There is a geometric interpretation of the second condition.

- Let $\overline{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$; the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ extends to $\overline{\mathbb{Q}}$ by $A\alpha :=\frac{a\alpha+b}{c\alpha+d}$ (these action sends $\overline{\mathbb{Q}}$ to itself). A cusp of $\mathbb{H}/G$ is an element of $\overline{\mathbb{Q}}/G$; in particular, if $G = \mathrm{SL}(2,\mathbb{Z})$ we have only one cusp which we can imagine to be ∞. In general, $\mathbb{H}/G$ can be compactified to a complete orbifold Riemann surface as $\overline{\mathbb{H}/G} = \mathbb{H}/G \cup \{\text{cusps}\}$.

- Let $\alpha \in \overline{\mathbb{Q}}$ and $A \in \mathrm{SL}(2,\mathbb{Z})$, with $A(\infty) = \alpha$. Let $l \geq 0$ such that $T^l \in A^{-1}GA$; then

  $$(f|_{k,A})|_{k,T^l} = f|_{k,AT^l} = f|_{k,A},$$

  that is, $f|_{k,A}$ is mapped to itself by $\tau \to \tau + l$. We fix l to be minimal with respect to his condition; this l is called *width* of the cusp. Now we can write $f|_{k,A} = \sum_{n\in\mathbb{Z}} a_n q^{n/l}$, and holomorphic at cusp α is equivalent to $a_n = 0$ for every $n < 0$.

- Geometrically, $\overline{\mathbb{H}/G}$ is a complex orbifold that has an obvious map φ to $\overline{\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})}$; this map is a branch cover of degree $[\mathrm{SL}(2,\mathbb{Z}) : G]$. The point ∞ in the target has as fiber the set of cusps in the source; moreover, the order of φ at a cusp is just its width (that is, at α, $q^{n/l}$ is a local coordinate).

6.3 EXAMPLE. Consider $G := \Gamma(2)/\{\pm I\}$; it can be proved that it is the free group $\langle \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right) \rangle$. A fundamental domain is represented in Figure 3. Its cusps then are $0, 1, \infty$; the width of ∞ is 2. As before, we define the set of modular forms to be $M_{k,G}$ with the subspace $S_{k,G}$ of cusp forms (that is, modular forms such that $f(\alpha) = 0$ for every cusp α). They are finite dimensional vector spaces and we can compute their dimensions.

6.4 EXAMPLE. The theta function $\Theta_{\mathbb{Z}^4}$ is $\sum_{n_1,\dots,n_4\in\mathbb{Z}} q^{\sum n_i^2}$. This is not even, so it is not a modular form; but a similar argument of the one did before shows that it is a modular form on some subgroup, precisely a modular form of weight 2 on $\Gamma^0(4)$.

6.5 COROLLARY.

1. *Every positive integer is the sum of four squares;*

2. $\left|\{n_1, \dots, n_4 \in \mathbb{Z} \mid \sum n_i^2 = n\}\right| = 8(\sum_{d|n, 4\nmid d} d)$.

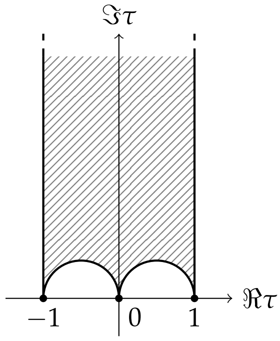


Figure 3: A fundamental domain of the action of G on $\mathbb{H}$.

Proof. The first statement is obvious; for the second, consider $8G_2(\tau) - 32G_2(4\tau)$; this is a modular form of weight 2 on $\Gamma^0(4)$. This is quite surprising since G_2 is not even; but we recall that $G_2^\star(\tau)$ is not holomorphic but transforms as a modular forms; so the one we are considering is just $8G_2^\star(\tau) - 32G_2^\star(4\tau)$. $\square$

7 Hecke theory

On modular forms there is an algebra of operators (the Hecke operators) such that there is a basis of simultaneus eigenvalues for the operators.

Recall that we have an isomorphism of vector spaces between:

- complex functions F of oriented lattices $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\Im(\omega_1/\omega_2) > 0$ such that $F(a\Lambda) = a^{-k}\Lambda$;
- holomorphic functions $f\colon \mathbb{H} \to \mathbb{C}$ such that $f(A\tau) = (c\tau + d)^{-k} f(\tau)$ for every $A \in \mathrm{SL}(2, \mathbb{Z})$.

In particular we associate to a morphism F the function $f(\tau) := F(\tau\mathbb{Z} \oplus \mathbb{Z})$ and to a function f the morphism such that $F(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) := \omega_2^{-k} f(\omega_1/\omega_2)$.

Let F be a lattice function of weight k; define the operators $T_n = T_{n,k}$ by

$$T_n F(\Lambda) := n^{k-1} \sum_{\Lambda' \subseteq \Lambda, [\Lambda:\Lambda'] = n} F(\Lambda').$$

These T_n have an interpretation as morphisms of moduli space of elliptic curves with additional level structure. Note anyway that $T_n F$ is a lattice function of weight k. Thus, denoting the corresponding function with $f\colon \mathbb{H} \to \mathbb{C}$, we define $T_n f(\tau) := T_n F(\tau\mathbb{Z} \oplus \mathbb{Z})$. Then for $T_n F$ to be a lattice function of weight k means that $T_n f(A\tau) = (c\tau + d)^{-k} T_n f$. After some computation we obtain a description in terms of τ: $T_n f(\tau) = n^{k-1} \sum_{A \in \Gamma\backslash\mathscr{M}_n} (c\tau + d)^{-k} f(A\tau)$. The summation indices means that A runs through a system of representatives of $\Gamma\backslash\mathscr{M}_n$, where $\mathscr{M}_n$ is the set of 2×2 matrices with entries in $\mathbb{Z}$ and determinant n, and $\mathrm{SL}(2, \mathbb{Z})$ acts on $\mathscr{M}_n$ by multiplication on the left.

If $f \in M_k$, then $T_n f$ is holomorphic on $\mathbb{H}$; plus, we already seen that it transforms as a modular forms; to check that $T_n f$ is a modular form, we need to prove that it is holomorphic at ∞; we do this writing down its q-development.

7.1 THEOREM.

1. *Let $f \in M_k$ with Fourier development $f(\tau) = \sum_{n\geq 0} c(n)q^n$; then*

$$T_n f(\tau) = \sum_{m\geq 0}\left(\sum_{d|n, d|m, d\geq 0} d^{k-1}\right) c(nm/d^2) q^m.$$

   In particular, $T_n f \in M_k$ and if f is a cusp form, then $T_n f$ is.

2. *T_n satisfies*

$$T_n T_m = \sum_{d|n, d|m, d\geq 0} d^{k-1} T_{nm/d^2};$$

   in particular, T_n and T_m commutes and if $(m,n) = 1$, $T_n T_m = T_{nm}$.

Proof. A system of representatives of $\Gamma\backslash\mathscr{M}_n$ is the set of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ such that $ad = n$ and $0 \leq b < d$. Then $T_n f(\tau) = n^{k-1}\sum_{a,d>0, ad=n}\sum_{b=0}^{d-1} d^{-k} f(\frac{a\tau+b}{d})$. Substituting the q-development of f we obtain

$$T_n f(\tau) = n^{k-1}\sum_{a,d>0, ad=n}\sum_{b=0}^{d-1} d^{-k}\sum_{m\geq 0} c(m) e^{2\pi\imath\mu^{a\tau+b/d}}.$$

Note that

$$\sum_{b=0}^{d-1} e^{2\pi\imath mb/d} = \begin{cases} 0 & d \nmid m \\ d & d \mid m \end{cases}$$

Now, observe that the second statement follows from the first by easy computations. $\square$

7.2 REMARK. Observe that $T_p T_{p^n} = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$ when p is prime; then if $n = p_1^{n_1}\cdots p_l^{n_l}$, then $T_n = T_{p_1^{n_1}}\cdots T_{p_l^{n_l}}$.

7.3 DEFINITION. The *Hecke operators* are the operators $T_n \colon M_k \to M_k$.

The Hecke operators are a set of commuting linear map; it is possible then to search for common eigenvectors, that is, modular forms f such that $T_n f = \lambda_n f$ for every $n \geq 1$.

7.4 DEFINITION. Let $f \in M_k$ be a common eigenvector of all T_n; assume $f(\tau) = \sum_{n\geq 0} a_n q^n$ with $a_n = 1$; then f is called a *Hecke form*.

The condition on $a_n = 1$ is to normalize the form. At first it appears that the parameter q is not so special; but it happens that the coefficients of the Fourier transform have a geometrical meaning; in particular, they are related to the eigenvalues.

7.5 corollary. *Let $f = \sum_{n\geq 0} a_n q^n$ be a Hecke form; then $T_n f = a_n f$ for all $n \geq 1$.*

7.6 corollary. *Let $f = \sum_{n\geq 0} a_n q^n$ be a Hecke form; then $a_n a_m = \sum_{d|n,d|m} d^{k-1} c(nm/d^2)$. In particular, if $(n,m) = 1$, then $a_n a_m = a_{nm}$.*

Proof. It is obvious by the formula of $T_n f(\tau)$. Since $T_n f(\tau) = \lambda_n f(\tau)$ and $a_1 = 1$, we can write

$$\lambda_n = \lambda_n a_1 = \sum_{d|n,d|1} d^{k-1} c(nm/d^2) = a_n. \qquad \square$$

7.7 example. The Eisenstein series G_k is a Hecke form for $k \geq 4$; so it is Δ.

7.8 corollary.

$$\tau(n)\tau(m) = \sum_{d|n,d|m} d^{11} \tau(nm/d^2).$$

7.9 theorem. *The Hecke forms form a basis for M_k for all k.*

8 L-series

Let $f := \sum_{n\geq 0} a_n q^n$ be a Hecke form; we can associate to it its *L-series*

$$L(f,s) := \sum_{n\geq 1} \frac{a_n}{n^s}$$

that converges absolutely and uniformly for $\Re s > k$ and is a holomorphic function for $\Re s > 0$. To prove these convergency results we need some machinery.

8.1 lemma. *Let $f := \sum_{n\geq 0} a_n q^n \in M_k$; then $a_n \in O(n^{k-1})$ (that is, $a_n/n^{k-1} \to 0$ as $n \to \infty$).*

8.2 corollary. *We can write $L(f,s)$ as an Euler product:*

$$L(f,s) = \prod_{p \text{ prime}} \frac{1}{1 - a_p p^{-s} + p^{k-1-2s}}.$$

Proof. From $a_n a_m = a_{nm}$ if $(n,m) = 1$, it follows that $a_{p_1^{n_1} \cdots p_l^{n_l}} = a_{p_1^{n_1}} \cdots a_{p_l^{n_l}}$; then $L(f,s) = \prod_{p \text{ prime}} \sum_{n\geq 0} a_{p^n} p^{-ns}$. We have to show that $\sum_{n\geq 0} a_{p^n} p^{-ns} = (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$. We know that $a_{p^{n+1}} - a_p a_{p^n} + p^{k-1} a_{p^{n-1}} = 0$ for p prime; if we multiply the series with $1 - a_p p^{-s} + p^{k-1-2s}$, we see that the constant term, with respect to $t := p^{-s}$ is 1, that the second term is 0 and by induction we get that all other coefficients are 0 using the previous relation. $\square$

8.3 example. We can write the Riemann Zeta function $\zeta(s) = \sum_{n\geq 1} 1/n^2$ as $\prod_{p \text{ prime}} 1/(1-p^{-s})$. We can compute $L(G_k, s)$; if p is a prime, $\sigma_{k-1}(p) = 1 + p^{k-1}$

and the denominator of the terms of the L-series is $1 + \sigma_{k-1}(p)p^{-s} + p^{k-1-2s} = (1 - p^{k-1-s})(1 - p^{-s})$ and it follows that

$$L(G_k, s) = \prod_{p \text{ primes}} \left(\frac{1}{1 - p^s}\right)\left(\frac{1}{1 - p^{k-1-s}}\right) = \zeta(s)\zeta(s - k + 1).$$

8.4 THEOREM. *If f is a Hecke form of weight k, then $L(f, s)$ has a meromorphic continuation to the whole $\mathbb{C}$ and satisfies some functional equation; if f is a cusp form, then $L(f, s)$ is an entire function; otherwise it has a simple pole at $s = k$.*

References

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7.