Algebraic stacks
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Contents

1 Introduction

Lecture 1 (2 ore)
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1.1 example. Let V be a finite dimensional vector space on an algebraically closed field K and $r \leq \dim V$; we can consider the grasmannian of r-dimensional vector subspace of V. This space is described as a set, but it could be naturally described as an algebraic variety. Moreover, it is a *fine moduli space*:

- the incidence variety $\Gamma(r, V) := \{(W, x) \mid W \in G(r, V) \wedge v \in W\}$ is a rank r vector subbundle of the trivial bundle $G(r, V) \times V$ over $G(r, V)$;
- if X is a scheme and $E \subseteq X \times V$ is a rank r vector subbundle, then there exists a unique $\varphi_E \colon X \to G(r, V)$ such that $E = \varphi_E^\star(\Gamma(r, V))$, or in other

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words such that the diagram

$$\begin{array}{ccc} E & \longrightarrow & \Gamma(r,V) \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[\varphi_E]{} & G(r,V) \end{array}$$

is cartesian.

1.2 exercise. Dimostrare che $G(r,V)$ è uno spazio di moduli fine.

1.3 remark. Come mappa di insiemi, φ_E manda x nel punto $[E_x]$ corrispondente al sottospazio vettoriale $E_x \subseteq \{x\} \times V$.

Assumendo sempre r e V fissi, consideriamo un'altra interpretazione. Sia $\gamma\colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$ il funtore controvariante definito in questo modo:

- se X è uno schema,

$$\gamma(X) := \{E \to X \mid E \text{ sottofibrato di rango } r \text{ di } X \times V\};$$

- se $\psi\colon X \to Y$ è un morfismo e $E_Y \subseteq Y \times V$ è un sottofibrato di rango r,

$$\gamma(\psi)(E_Y) := \psi^\star(E_Y) = (\psi \times \mathrm{id})^{-1}(E_Y).$$

Con questo linguaggio, il fatto che $G(r,V)$ è uno spazio di moduli fine si esprime dicendo che γ è naturalmente isomorfo al funtore di Yoneda $\mathrm{h}_{G(r,V)}$. In particolare, a $\mathrm{id} \in \mathrm{Aut}(G(r,V))$ corrisponde il fibrato $\Gamma(r,V) \to G(r,V)$.

1.4 example. Se X è una varietà proiettiva (volendo anche liscia), consideriamo $\mathrm{Pic}(X)$, l'insieme dei fibrati lineari su X modulo isomorfismo; $\mathrm{Pic}(X)$ ha una struttura di spazio topologico con, in generale, infinite componenti connesse e ogni componente ha una struttura di varietà algebrica. Inoltre, la struttura di gruppo su $\mathrm{Pic}(X)$ (data dal prodotto tensore) è compatibile con la struttura di varietà algebrica. Grazie a questo si dimostra che tutte le componenti connesse sono isomorfe tra loro e quindi si può considerare solamente la componente connessa dell'identità, $\mathrm{Pic}^0(X)$.

Come costruire il funtore corrispondente a $\mathrm{Pic}^0(X)$, per ottenere una situazione analoga alla precedente?

1.5 definition. Sia X una varietà proiettiva liscia; si definisce il funtore controvariante $\pi\colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$ mediante:

- se S è uno schema,

$$\pi(S) := \{L \mid L \text{ fibrato lineare su } X \times S\}/\sim,$$

dove $L \sim L'$ sono equivalenti se e solo se esiste $M \in \mathrm{Pic}(S)$ tale che

$L' \cong L \otimes p_S^\star M$, dove $p_S\colon X \times S \to S$ è la proiezione[1];

- se $\varphi\colon S' \to S$ e $L \in \operatorname{Pic}(X \times S)$,

$$\pi(\varphi)(L) := \varphi^\star L;$$

si può verificare che questo passa alla relazione d'equivalenza.

Non dimostreremo il seguente teorema.

1.6 THEOREM. *Esiste unica una struttura di varietà algebrica (con infinite componenti connesse) su* $\operatorname{Pic}(X)$ *ed esiste un fibrato lineare* $\mathscr{L}$, *detto* fibrato di Poincaré *su* $X \times \operatorname{Pic}(X)$ *tale che* $\pi \to \mathrm{h}_{\operatorname{Pic}(X)}$ *è una equivalenza naturale e* $\mathscr{L}$ *corrisponde a* $\mathrm{id} \in \operatorname{Aut}(\operatorname{Pic}(X))$; *inoltre* $\mathscr{L}$ *è unico a meno di pullback tramite la proiezione* $X \times \operatorname{Pic}(X) \to \operatorname{Pic}(X)$.

Possiamo riformulare il teorema nel modo seguente. Il funtore π è rappresentato da una varietà algebrica liscia (con infinite componenti connesse); da questo segue che i K-punti di $\operatorname{Pic}(X)$, per definizione in corrispondenza biunivoca con $\operatorname{Mor}(\operatorname{Spec} K, \operatorname{Pic}(X))$, per la rappresentabilità sono in corrispondenza biunivoca anche con $\pi(\operatorname{Spec} K)$; questo però corrisponde a $\operatorname{Pic}(X)$, a priori a meno di pullback di fibrati lineari sul punto, che però sono solo banali.

Perché abbiamo la complicazione della tensorizzazione per un pullback? Questo viene dal fatto che due fibrati lineari possono avere diversi modi per essere isomorfi, cioè dal fatto che, al contrario dei sottospazi vettoriali di V, un fibrato lineare può avere un gruppo di automorfismi non banali.

La prima apparizione degli stack è all'inizio degli anni '60, in francese, con il nome di *champs*, da parte di Giraud, studente di Grothendieck. Ma l'introduzione degli stack come oggetti geometrici è dovuta a Deligne e Mumford, alla fine degli anni '60, quando si posero il seguente problema.

1.7 PROBLEM. Sia $g \geq 2$ un intero; si vuole dare una struttura di varietà algebrica a M_g, lo spazio delle curve lisce proiettive di genere g modulo isomorfismo; se possibile come spazio di moduli fine (cioè in modo che sia la rappresentazione di un funtore adatto).

Perché di genere maggiore o uguale a 2? Perché le curve con tali proprietà e di genere 0 sono solo $\mathbb{P}^1$, mentre quelle di genere 1 sono curve ellittiche, studiate da molto tempo e sufficientemente comprese. Il primo approccio a questo problema è proprio di Riemann, che dimostrò che per descrivere una curva di genere g sono necessari $3g-3$ parametri che chiamò *moduli*.

1.8 EXAMPLE. Le curve di genere 2 sono tutte iperellittiche; in particolare, se C ha genere 2 esiste un unica mappa $C \to \mathbb{P}^1$ due a uno, che è ramificata in 6 punti. Viceversa, un tale morfismo ramificato determina una curva di genere 2. Quindi le curve di genere 2 sono in corrispondenza con i sottoinsiemi di 6 punti di $\mathbb{P}^1$ modulo automorfismi di $\mathbb{P}^1$; se questi punti fossero ordinati,

[1] In tal modo, $L'|_{X\times\{s\}} \cong L|_{X\times\{s\}} \otimes p_S^\star M|_{X\times\{s\}} = L|_{X\times\{s\}}$, cioè si considerano equivalenti due fibrati se si restringono sulla fibra allo stesso oggetto.

si potrebbero esaurire gli automorfismi fissando $p_1 = 0$, $p_2 = 1$, $p_3 = \infty$ e il quoziente avrebbe dimensione 3. Non essendo ordinati, c'è un'altra azione di S_6 da tenere in considerazione, che però non cambia la dimensione del quoziente.

1.9 example. In genere 3, abbiamo le curve iperellittiche (con morfismo su $\mathbb{P}^1$ ramificato in 8 punti) e le curve non iperellittiche, che ammettono un unico morfismo su $\mathbb{P}^2$ (modulo automorfismo di $\mathbb{P}^2$) su una curva di grado 4. Le prime hanno 5 moduli, calcolati nel modo visto prima; di conseguenza le seconde devono averne 6.

1.10 exercise. Dimostrare che le curve di genere 3 non iperellittiche hanno 6 moduli.

1.11 definition. Sia $\mu_g \colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$ il funtore definito in questo modo:

- dato uno schema S,

$$\mu_g(S) := \left\{ p \colon C \to S \;\middle|\; \begin{array}{l} p \text{ liscio, proiettivo, con} \\ C_s := p^{-1}(s) \text{ una curva di} \\ \text{genere } g \text{ per ogni } s \in S \end{array} \right\} \Big/ \sim ,$$

  dove $p \sim p'$ se esiste un diagramma di questo tipo:

$$\begin{tikzcd} C \arrow[r, "\sim"] \arrow[d, "p"'] & C' \arrow[d, "p'"] \\ S \arrow[r, equal] & S'. \end{tikzcd}$$

- dato un morfismo $\varphi \colon S' \to S$ e una famiglia $p \colon C \to S$,

$$\mu_g(\varphi)(p) \colon C \times_S S' \to S';$$

  questa nuova famiglia mantiene tutte le proprietà richieste.

In particolare, osserviamo che $\mu_g(\operatorname{Spec} K) = M_g$.

1.12 problem. Esiste una struttura di varietà algebrica su M_g tale che μ_g e h_{M_g} siano naturalmente equivalenti? In altre parole, μ_g è rappresentabile?

La risposta a questa domanda è nella seguente proposizione, che verrà dimostrata in seguito.

1.13 proposition. *Se si definisce μ_g^0 come μ_g, con la condizione aggiuntiva che ogni fibra sia una curva rigida (senza automorfismi non banali), allora μ_g^0 è rappresentabile con una varietà liscia M_g^0, quasi proiettiva, connessa, di dimensione $3g-3$, a meno che $g = 2$ (perché tutte le curve di genere 2, essendo iperellittiche, hanno almeno un'involuzione).*

1.14 example. Per $g = 3$, si considera U, l'insieme delle quartiche di $\mathbb{P}^2$ lisce e rigide (cioè, i cui soli automorfismi siano indotti da automorfismi di $\mathbb{P}^2$); allora $U/\mathbb{P}\,\mathrm{GL}(3,K)$ è M_3^0, dato che l'azione di $\mathbb{P}\,\mathrm{GL}(3,K)$ è senza punti fissi. Com'è descritta la curva universale? Sia $\Gamma_U \subseteq U \times \mathbb{P}^2$ la varietà d'incidenza, con punti (C,x) con $x \in C$; allora $\Gamma_U/\mathbb{P}\,\mathrm{GL}(3,K) \to U/\mathbb{P}\,\mathrm{GL}(3,K)$ è la famiglia universale, in quanto le sue fibre sono esattamente le fibre di $\Gamma_U \to U$. Se si cerca di inserire anche le quartiche non rigide, nel quozientare si incontrano problemi a causa degli stabilizzatori non banali.

Come si dimostra che un funtore non è rappresentabile? Vediamo un'esempio di dimostrazione per il funtore μ_g.

1.15 PROPOSITION. *Il funtore μ_g non è rappresentabile.*

Proof. Abbiamo visto che i problemi nascono dalle curve con automorfismi; sia quindi C_0 una curva liscia di genere g con un automorfismo φ diverso dall'identità. Per fissare le idee, supponiamo $\varphi^2 = \mathrm{id}$ (in realtà è sempre possibile trovare una curva con un'involuzione, dato che in ogni genere ci sono curve iperellittiche). Consideriamo $S := \mathbb{A}^1 \setminus \{0\}$ e la famiglia banale $p\colon C_0 \times S \to S$; p è il pullback di $C_0 \to \operatorname{Spec} K$ via l'unica mappa $S \to \operatorname{Spec} K$. Se μ_g fosse rappresentabile, p dovrebbe necessariamente corrispondere a un morfismo costante $S \to M_g$.

Siano ora $\alpha\colon S \to S$ l'involuzione data da $\alpha(t) := -t$ e $S' := S/\alpha$, ancora isomorfo a S con coordinata $s := t^2$. Sia inoltre $C' := (C_0 \times S)/(\varphi,\alpha)$, dove $(\varphi,\alpha)(x,t) = (\varphi(x), -t)$. Quindi il morfismo p induce un morfismo $p'\colon C' \to S'$, con le stesse fibre; in particolare si dimostra che $C_0 \times S \cong C' \times_{S'} S$. Allora tutte le fibre di p' sono isomorfe a C_0, perciò se il funtore fosse rappresentabile, p' dovrebbe essere il pullback di una famiglia universale su M_g tramite una mappa costante, e C' dovrebbe essere un prodotto, ma non è così. $\square$

In questo caso, si dice che $C' \to S'$ è una famiglia *isotriviale*, cioè tutte le fibre sono isomorfe ma globalmente non è un prodotto.

Dimostrazione alternativa. Sia $T := \mathbb{A}^1$ e consideriamo $C'' := (C_0 \times T)/\sim$ dove $(x,t) \sim (x',t')$ se sono uguali o $x' = \varphi(x)$ e $\{t,t'\} = \{-1,1\}$. In altre parole, si identificano le fibre su -1 e 1 e si incollano rovesciate tramite φ. Si dimostra che C'' ha una struttura di varietà algebrica.

Sia ora $T'' := T/(-1 = +1)$; T'' si realizza in $\mathbb{A}^2$ tramite una cubica nodata. Ancora, $C'' \to T''$ è una famiglia isotriviale. $\square$

1.16 EXERCISE. Dimostrare che C' non è un prodotto; si può assumere che $\mathrm{Aut}(C_0) \cong \{\mathrm{id}, \alpha\}$.

Il problema si pone allo stesso modo con i fibrati: tutti i fibrati dello stesso rango sono localmente isomorfi, ma questo non si estende in generale a un isomorfismo globale; per avere uno spazio di moduli fine, ovvero se si vuole considerare un fibrato come un morfismo in un qualche spazio, bisogna anche considerare gli isomorfismi dei morfismi, che nel caso della topologia sono le omotopie.

Cos'è andato storto nella definizione di μ_g? L'idea di uccidere tutti gli isomorfismi considerandoli tutti allo stesso modo. Se μ_g fosse naturalmente isomorfo a h_{M_g}, allora $\mu_g(S)$ sarebbe in bîezione con $\mathrm{Mor}(S, M_g)$; per non dimenticare gli isomorfismi, si deve definire $\mu_g(S)$ come un oggetto che ricordi sia un insieme di famiglie di curve di genere g con le proprietà richieste, ma anche gli isomorfismi tra queste famiglie. Un oggetto tale è un *gruppoide*, cioè una categoria in cui ogni morfismo è un isomorfismo.

1.17 example. Siano $\mathfrak{C}$ e $\mathfrak{C}'$ due categorie, $F, G\colon \mathfrak{C} \to \mathfrak{C}'$ due funtori covarianti; un funtore manda oggetti in oggetti e morfismi in morfismi; se abbiamo una trasformazione naturale $\nu\colon F \to G$, questa manda un oggetto $x \in \mathfrak{C}$ in un morfismo $\nu(x)\colon F(x) \to G(x)$, mentre manda un morfismo $f\colon x \to y$ in un diagramma commutativo in $\mathfrak{C}'$:

$$\begin{tikzcd} F(x) \arrow[r, "\nu(x)"] \arrow[d, "F(f)"'] & G(x) \arrow[d, "G(f)"] \\ F(y) \arrow[r, "\nu(y)"'] & G(y). \end{tikzcd}$$

La trasformazione naturale ν è un'equivalenza naturale se $\nu(x)$ è un isomorfismo per ogni x. In particolare, se $\mathfrak{C}'$ è un gruppoide, ogni trasformazione naturale $\nu\colon F \to G$ è un'equivalenza naturale.

1.18 corollary. *I gruppoidi hanno una struttura di* 2*-categoria, in cui ci sono:*

- *gli oggetti: i gruppoidi stessi;*
- *i morfismi: i funtori covarianti tra gruppoidi;*
- *i* 2*-morfismi: le equivalenze naturali;*

i 2*-morfismi, o morfismi tra morfismi, sono tutti invertibili; in particolare, se X e Y sono gruppoidi, allora* $\mathrm{Mor}(X, Y)$*, la categoria con oggetti i funtori e morfismi le trasformazioni naturali, è ancora un gruppoide.*

L'idea è prendere la definizione di schema, scriverla in un modo in cui sia evidente che i morfismi formino un insieme, sostituire i gruppoidi agli insiemi e aggiustare le cose.

1.19 notation. Siano $\mathfrak{C}$ e $\mathfrak{C}'$ gruppoidi, $F, G\colon \mathfrak{C} \to \mathfrak{C}'$ funtori, ν una equivalenza naturale tra F e G (denotata $\nu\colon F \Rightarrow G$); questa situazione è descritta dal seguente diagramma:

$$\begin{tikzcd} \mathfrak{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathfrak{C}' \arrow[from=U, to=D, Rightarrow, "\nu"] \end{tikzcd}$$

si dice che il diagramma è 2-commutativo.

1.20 exercise. Costruire la composizione

$$
\begin{tikzcd}
\mathfrak{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathfrak{C}' \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathfrak{C}''
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathfrak{C} \arrow[rr, bend left, "F_2\circ F_1"] \arrow[rr, bend right, "G_2\circ G_1"'] \arrow[rr, phantom, "\Downarrow\mu\circ\lambda"] & & \mathfrak{C}''.
\end{tikzcd}
$$

1.21 exercise. Costruire la composizione

$$
\begin{tikzcd}[row sep=large]
\mathfrak{C} \arrow[r, bend left=60, "F_1"] \arrow[r] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\lambda"] \arrow[r, phantom, shift right=3ex, "\Downarrow\mu"] & \mathfrak{C}'
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathfrak{C} \arrow[r, bend left=60, "F_1"] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, "\Downarrow\mu\circ\lambda"] & \mathfrak{C}'
\end{tikzcd}
$$

1.22 exercise. Costruire la composizione

$$
\begin{tikzcd}
\mathfrak{A} \arrow[r] \arrow[d] & \mathfrak{B} \arrow[d] \arrow[dl] \\
\mathfrak{C} \arrow[r] & \mathfrak{D}
\end{tikzcd}
\;(\Swarrow\nu,\ \Nearrow\mu)\;
\rightsquigarrow
\begin{tikzcd}
\mathfrak{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow\mu\circ\nu"] & \mathfrak{B} \arrow[d] \\
\mathfrak{C} \arrow[r] & \mathfrak{D}
\end{tikzcd}
$$

2 Gruppoidi

Lecture 2 (2 ore)
January 14^{th}, 2009

2.1 definition. Sia X uno spazio topologico; il *gruppoide fondamentale* è

$$
\pi_1(X) := \begin{cases} x \in \mathrm{Obj}\,(\pi_1(X)) & \Leftrightarrow \quad x \in X, \\ [f] \in \mathrm{Mor}\,(x,y) & \Leftrightarrow \quad \begin{cases} f\colon [0,1] \to X \text{ continua}, \\ f(0) = p, f(1) = q. \end{cases} \end{cases}
$$

dove la classe di f è presa modulo omotopia relativa a $\{0,1\}$.

2.2 exercise.

1. Il gruppoide fondamentale di X è una categoria e in particolare un gruppoide;
2. dati $p, q \in X$, p è isomorfo a q se e solo se p e q sono nella stessa componente connessa per archi;
3. dato $p \in X$, $\mathrm{Aut}(X) = \pi_1(X, x)$.

Osserviamo che i gruppi fondamentali con punti base x e y sono isomorfi se x e y sono isomorfi nel gruppoide $\pi_1(X)$; questa è una proprietà più generale, come mostra il seguente esercizio.

2.3 exercise. Sia G un gruppoide; se $x, y \in \mathrm{Obj}(G)$ sono isomorfi, allora esiste un isomorfismo $\mathrm{Aut}(x) \to \mathrm{Aut}(y)$, canonico a meno di coniugio con un automorfismo di y.

In generale, gli oggetti di una categoria (ovvero di un gruppoide) sono una classe; nel seguito ignoreremo completamente di considerare questa complicazione; in particolare, lavoreremo solo su gruppoidi *piccoli* (dove la classe degli oggetti è un insieme) e usando l'assioma della scelta.

2.4 definition. Un funtore $F\colon G_1 \to G_2$ è una *equivalenza* se esiste $G\colon G_2 \to G_1$ tale che $F \circ G$ e $G \circ F$ sono naturalmente equivalenti all'identità.

2.5 exercise. Il funtore F è un'equivalenza se e solo se sono vere entrambe le seguenti proposizioni:

1. F è *pienamente fedele*, cioè per ogni $x, x' \in G_1$,

   $$F\colon \operatorname{Mor}(x, x') \to \operatorname{Mor}(F(x), F(x'))$$

   è biunivoca;

2. F è *essenzialmente suriettivo*, cioè per ogni $y \in G_2$ esiste $x \in G_1$ tale che $F(x)$ è isomorfo a y.

Per mostrare che le due condizioni implicano che F sia un'equivalenza, è necessario l'assioma della scelta.

2.6 theorem. *Sia G un gruppoide e sia H un sottogruppoide tale che* $\operatorname{Obj}(H)$ *contenga esattamente un oggetto per ogni classe di isomorfismo di oggetti di G; richiediamo inoltre che H sia un sottogruppoide* pieno, *cioè che per ogni $x, y \in H$,* $\operatorname{Mor}_H(x, y) = \operatorname{Mor}_G(x, y)$. *Allora il funtore inclusione $H \to G$ è un'equivalenza.*

Proof. Usiamo il criterio dell'esercizio 2.5: l'inclusione è essenzialmente suriettiva e pienamente fedele per definizione. □

2.7 definition. Un gruppoide in cui tutti i morfismi siano automorfismi si dice *disconnesso*.

2.8 remark. Ogni gruppoide G ammette un sottogruppoide pieno H che sia disconnesso, grazie all'assioma della scelta. Ovviamente in generale H non è univocamente determinato.

2.9 definition. Possiamo associare a un insieme S il gruppoide

$$S := \begin{cases} x \in \operatorname{Obj}(S) & \Leftrightarrow \quad x \in S, \\ f \in \operatorname{Mor}(x, y) & \Leftrightarrow \quad x = y, f = \mathrm{id}_x\,. \end{cases}$$

2.10 definition. Se G è un gruppoide, $\pi_0(G)$ è definito come l'insieme delle classi di equivalenza di oggetti di G modulo gli isomorfismi in G.

2.11 exercise. Siano G un gruppoide, e S un insieme; allora $\operatorname{Mor}(\pi_0(G), S)$ è in corrispondenza biunivoca con $\operatorname{Fun}(G, S)$, dove S è visto come gruppoide.

2.12 definition. Un oggetto x in un gruppoide è *rigido* se $\operatorname{Aut}(x) = \{\mathrm{id}\}$; un gruppoide è *rigido* se ogni oggetto è rigido.

2.13 EXERCISE.

1. Se x è un oggetto rigido, ogni oggetto a lui isomorfo è rigido;

2. se G è un gruppoide rigido, è equivalente al gruppoide associato a un insieme, in particolare è equivalente a $\pi_0(G)$.

2.14 EXERCISE. Sia $F\colon G_1 \to G_2$ un funtore tra gruppoidi; se G_2 è rigido, allora l'unica equivalenza naturale $F \Rightarrow F$ è l'identità.

2.15 EXERCISE. Siano X un insieme e G un gruppo che agisce (a sinistra) su X; possiamo costruire un gruppoide che per adesso chiameremo $[X/G]$ in questo modo:

$$[X/G] := \begin{cases} x \in \mathrm{Obj}\,([X/G]) & \Leftrightarrow \quad x \in X, \\ g \in \mathrm{Mor}\,(x,y) & \Leftrightarrow \quad g \in G, g \cdot x = y. \end{cases}$$

Definire la composizione e dimostrare che questo è un gruppoide; dimostrare inoltre che $\mathrm{Mor}([X/G]) := \bigsqcup_{x,y \in X \times X} \mathrm{Mor}(x,y) = G \times X$. Trovare una bîezione naturale

$$\left\{ \begin{matrix} \text{gruppoidi} \\ \text{rigidi} \end{matrix} \right\} \leftrightarrow \left\{ (X,R) \;\middle|\; \begin{matrix} \text{X insieme,} \\ R \subseteq X \times X \text{ relazione d'equivalenza} \end{matrix} \right\}.$$

Possiamo quindi immaginare un gruppoide sia come un'estensione di un insieme, di un gruppo o di una relazione d'equivalenza.

2.16 DEFINITION. Un diagramma commutativo

$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

in una categoria $\mathfrak{C}$ si dice *cartesiano* se per ogni diagramma commutativo del tipo

$$\begin{tikzcd} V' \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & V \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

esiste unico un morfismo $V' \to V$ che commuta con il diagramma.

In particolare, se $\mathfrak{C} = \mathfrak{Sets}$, dati $f\colon X \to Z$ e $g\colon Y \to Z$, possiamo definire il loro *prodotto fibrato* come

$$X \times_Z Y := \{(x,y) \in X \times Y \mid f(x) = g(y)\};$$

è ben noto che il prodotto fibrato con le due proiezioni rende il diagramma cartesiano; inoltre ogni insieme che rende cartesiano il diagramma è isomorfo a $X \times_Z Y$, grazie alla proprietà universale.

Vogliamo trovare un'analoga proprietà universale per la cartesianità nel contesto delle 2-categorie (o più in particolare, per i gruppoidi).

2.17 definition. Siano $f\colon X \to Z$ e $g\colon Y \to Z$ morfismi di gruppoidi; si definisce il *prodotto fibrato*

$$X \times_Z Y := \begin{cases} (x, y, \varphi) \in \operatorname{Obj}(X \times_Z Y) \quad \Leftrightarrow \quad \begin{cases} x \in \operatorname{Obj}(X), y \in \operatorname{Obj}(Y), \\ \varphi \in \operatorname{Mor}_Z(f(x), g(y)), \end{cases} \\ (\alpha, \beta) \in \operatorname{Mor}((x, y, \varphi), (x', y', \varphi')) \quad \Leftrightarrow \quad \begin{cases} \alpha \in \operatorname{Mor}_X(x, x'), \\ \beta \in \operatorname{Mor}_Y(y, y'), \\ \begin{tikzcd} f(x) \arrow[r, "\varphi"] \arrow[d, "f(\alpha)"'] \arrow[dr, phantom, "\circlearrowleft"] & g(y) \arrow[d, "g(\beta)"] \\ f(x') \arrow[r, "\varphi'"'] & g(y'). \end{tikzcd} \end{cases} \end{cases}$$

2.18 lemma. *Il prodotto fibrato ha una naturale struttura di gruppoide.*

Proof. L'identità dell'oggetto (x, y, φ) è data da $(\mathrm{id}_x, \mathrm{id}_y)$; la composizione di

$$(x, y, \varphi) \xrightarrow{(\alpha,\beta)} (x', y', \varphi') \xrightarrow{(\alpha',\beta')} (x'', y'', \varphi'')$$

è $(\alpha' \circ \alpha, \beta' \circ \beta)$, ben definito in quanto

$$\begin{tikzcd} f(x) \arrow[r, "\varphi"] \arrow[d, "f(\alpha)"] \arrow[dd, bend right, "f(\alpha'\circ\alpha)"'] & g(y) \arrow[d, "g(\beta)"'] \arrow[dd, bend left, "g(\beta'\circ\beta)"] \\ f(x') \arrow[r, "\varphi'"] \arrow[d, "f(\alpha')"] & g(y') \arrow[d, "g(\beta')"'] \\ f(x'') \arrow[r, "\varphi''"] & g(y''). \end{tikzcd}$$

L'inverso di (α, β) è $(\alpha^{-1}, \beta^{-1})$. □

2.19 definition. Definiamo i funtori $p_1\colon X \times_Z Y \to X$ e $p_2\colon X \times_Z Y \to Y$ ponendo

$$\begin{aligned} p_1(x, y, z) &:= x & \qquad p_1(\alpha, \beta) &:= \alpha, \\ p_2(x, y, z) &:= y & \qquad p_2(\alpha, \beta) &:= \beta. \end{aligned}$$

Osserviamo che il diagramma

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

in generale non commuta, dato che $f \circ p_1(x, y, \varphi) = f(x)$ mentre $g \circ p_2(x, y, \varphi) = g(y)$.

2.20 LEMMA. *Esiste un'equivalenza naturale* $\varrho\colon p_1 \circ f \Rightarrow p_2 \circ g$ *data da*

$$\varrho(x, y, \varphi) := \varphi\colon f(x) \to g(y).$$

Proof. Immediato dalla definizione di $X \times_Z Y$. $\square$

L'ultima cosa da fare è adattare la proprietà universale di diagramma cartesiano alle 2-categorie e dimostrare che il prodotto fibrato di gruppoidi la soddisfa.

2.21 THEOREM. *Sia*

$$\begin{tikzcd} V \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, "\overset{\eta}{\Nearrow}"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

un diagramma 2*-commutativo; allora esiste unica* $h\colon V \to X \times_Y Z$ *tale che*

$$\begin{tikzcd} V \arrow[dr, "h"] \arrow[drr, bend left, "q_1"] \arrow[ddr, bend right, "q_2"'] & & \\ & X \times_Z Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{\varrho}{\Nearrow}"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

$$\text{(2-celle: } \overset{\mathrm{id}}{\Nearrow},\ \overset{\mathrm{id}}{\Nearrow}\text{)}$$

è 2*-commutativo; in particolare*

- $q_1 = h \circ p_1$,
- $q_2 = h \circ p_2$,
- ϱ *induce* η.

Proof. Il fatto che η sia una trasformazione naturale che rende il primo diagramma 2-commutativo si traduce in questo modo:

$$\forall v \in \mathrm{Obj}(V), \eta(v)\colon f(q_1(v)) \to g(q_2(v)).$$

Dobbiamo definire $h(v)$: l'unica scelta plausibile è $h(v) := (q_1(v), q_2(v), \eta(v))$, mentre per un morfismo $\psi\colon v_1 \to v_2$, $h(\psi) := (q_1(\psi), q_2(\psi))$. Rimangono da verificare che queste siano buone definizioni, in particolare che $h(\psi)$ sia un morfismo e che il diagramma

$$\begin{tikzcd} f(q_1(v_1)) \arrow[r, "\eta(v_1)"] \arrow[d, "f(q_1(\psi))"'] & g(q_2(v_1)) \arrow[d, "g(q_2(\psi))"] \\ f(q_1(v_2)) \arrow[r, "\eta(v_2)"'] & g(q_2(v_2)) \end{tikzcd}$$

commuta, cosa che è evidente dal fatto che η è un'equivalenza naturale. L'ultima condizione, che ϱ induce η, si traduce semplicemente dicendo che l'ultima componente di $h(v)$ deve essere $\eta(v)$, come definito in precedenza. □

Abbiamo cambiato la proprietà universale, ma in un certo senso non molto: i 2-morfismi in alto e a sinistra devono essere l'identità. In realtà quella esposta è una versione edulcorata della definizione di diagramma 2-cartesiano.

2.22 definition. Un diagramma 2-commutativo

$$\begin{tikzcd} V \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[dr, phantom, "\eta \nearrow"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

si dice 2-*cartesiano* se il funtore $h\colon V \to X \times_Z Y$ definito nel Teorema 2.21 è un'equivalenza di gruppoidi.

2.23 proposition. *Un diagramma* 2*-cartesiano ha la seguente proprietà universale: per ogni diagramma* 2*-commutativo*

$$\begin{tikzcd} V' \arrow[r, "q'_1"] \arrow[d, "q'_2"'] \arrow[dr, phantom, "\eta' \nearrow"] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

esiste un funtore $h'\colon V' \to V$ ed esistono dei 2*-morfismi $\vartheta'_1\colon q_1 \circ h' \Rightarrow q'_1$ e $\vartheta'_2\colon q_2 \circ$*

$h' \Rightarrow q'_2$ tale che

$$\begin{tikzcd}
V' \arrow[dr, "h'"] \arrow[drr, bend left, "q'_1"] \arrow[ddr, bend right, "q'_2"'] & & \\
& V \arrow[r, "q_1"] \arrow[d, "q_2"'] \arrow[u, phantom, "\vartheta'_1 \Rightarrow"] \arrow[l, phantom, "\vartheta'_2 \Rightarrow"] \arrow[dr, phantom, "\varrho \Rightarrow"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

induce η'. Inoltre $(h', \vartheta'_1, \vartheta'_2)$ sono unici a meno di 2-isomorfismo, cioè dati $(h'', \vartheta''_1, \vartheta''_2)$ con le stesse proprietà, esiste unico $\delta\colon h' \Rightarrow h''$ tale che il diagramma (visto prima da sopra e poi da sotto)

$$\begin{tikzcd}
V' \arrow[d, "q'_1"] \arrow[d, bend right=80, "q_1 \circ h''"'] \arrow[d, bend left=80, "q_1 \circ h'"] \arrow[d, phantom, shift right=6ex, "\overset{\vartheta'_1}{\Longrightarrow}"] \arrow[d, phantom, shift left=6ex, "\overset{\vartheta''_1}{\Longleftarrow}"] \\
X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V' \arrow[d, bend right=80, "q_1 \circ h'"'] \arrow[d, bend left=80, "q_1 \circ h''"] \arrow[d, phantom, "\overset{q_1(\delta)}{\Longleftarrow}"] \\
X
\end{tikzcd}$$

sia 2-commutativo, e allo stesso modo per ϑ'_2 e ϑ''_2.

È possibile usare la semplificazione vista all'inizio in quanto si dimostra che tra questi dati ne esiste sempre uno con $\vartheta'_1 = \mathrm{id}$ e $\vartheta'_2 = \mathrm{id}$.

2.24 EXERCISE. Se X, Y, Z sono insiemi e X', Y', Z' sono gli stessi insiemi visti come gruppoidi, allora $X' \times_{Z'} Y'$ è l'insieme $X \times_Z Y$ visto come gruppoide

Soluzione. Sia $(x, y, \varphi) \in X' \times_{Z'} Y'$, cioè $x \in X$, $y \in Y$ e $\varphi\colon f(x) \to g(y)$; allora $f(x) = g(y)$ e $\varphi = \mathrm{id}$, dato che Z è un insieme; mentre un morfismo da $(x, y, \mathrm{id}_{f(x)}$ a $(x', y', \mathrm{id}_{f(x')})$ può essere solo $(\mathrm{id}_x, \mathrm{id}_y)$ se $x = x'$ e $y = y'$. □

2.25 EXERCISE. Il prodotto fibrato è stabile, modulo equivalenza, sostituendo X, Y, Z con gruppoidi equivalenti. Per esempio, se $a\colon X' \to X$ è un'equivalenza naturale, allora a induce un'equivalenza $X' \times_Z Y \to X \times_Z Y$; lo stesso per $b\colon Y' \to Y$ e $c\colon Z \to Z'$.

2.26 PROPOSITION. *Sia G un gruppoide che agisce su un insieme X. Denotiamo X' il gruppoide associato a X; definiamo $\pi\colon X' \to [X/G]$ ponendo*

$$\pi(x) := x, \qquad\qquad \pi(\mathrm{id}_x) := \mathrm{id}_x\,,$$

$p_2\colon G \times X' \to X'$ con $p_2(g, x) := x$ e $a\colon G \times X' \to X'$ con $a(g, x) := g \cdot x$. Il

diagramma

$$\begin{tikzcd} G \times X' \arrow[r, "a"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\overset{\eta}{\nearrow}"] & X' \arrow[d, "\pi"] \\ X' \arrow[r, "\pi"'] & {[X/G]} \end{tikzcd}$$

è naturalmente 2*-commutativo tramite un* 2*-morfismo* η*, che inoltre rende il diagramma cartesiano.*

Proof. Sia $(g, x) \in \mathrm{Obj}(G \times X')$; allora $(\pi \circ p_2)(g, x) = x$, mentre $(\pi \circ p_1)(g, x) = g \cdot x$; l'unico modo per definire η è ponendo $\eta(g, x) := g \in \mathrm{Mor}_{[X/G]}(x, g \cdot x)$.

Sia ora Y il prodotto fibrato $X' \times_{[X/G]} X'$; i suoi oggetti sono triple (x_1, x_2, g) con $x_1, x_2 \in X$ e $g \in G$ tale che $g \cdot x_1 = x_2$. I morfismi tra (x_1, x_2, g) e (x_1', x_2', g') sono coppie di morfismi $x_1 \to x_1'$ e $x_2 \to x_2'$, quindi necessariamente, dato che X è un insieme, non ci sono morfismi se $x_1 \neq x_1'$ o $x_2 \neq x_2'$; inoltre, perché $(\mathrm{id}_{x_1}, \mathrm{id}_{x_2})$ sia davvero un morfismo, il diagramma

$$\begin{tikzcd} x_1 \arrow[r, "g"] \arrow[d, "\mathrm{id}_{x_1}"'] & x_2 \arrow[d, "\mathrm{id}_{x_2}"] \\ x_1 \arrow[r, "g"'] & x_2 \end{tikzcd}$$

deve commutare. In definitiva, gli unici morfismi sono le identità se le due triple sono uguali, cioè Y è un insieme. L'applicazione $G \times X' \to Y$, indotta dal fatto che Y è un prodotto fibrato, è data da $(g, x) \mapsto (x, g \cdot x, g)$, come applicazione tra insiemi, che è chiaramente biunivoca (l'inversa dimentica il secondo elemento della tripla). □

Ritornando agli insiemi, consideriamo l'azione di un gruppo G su un insieme X. Possiamo costruire un insieme quoziente X/G e un diagramma commutativo

$$\begin{tikzcd} G \times X \arrow[r, "a"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & X/G \end{tikzcd}$$

dove a è l'azione e p_2 la proiezione.

2.27 exercise. Il diagramma è cartesiano se e solo se l'azione di G è libera (cioè se ogni stabilizzatore è banale).

TODO

2.28 remark. Sia $x \in \mathrm{Obj}([X/G])$; allora $\mathrm{Aut}(x) = \mathrm{Stab}_G(x)$. Quindi considerando i gruppoidi, tutte le azioni di gruppo si comportano come un'azione libera.

2.29 definition. Sia G un gruppoide; il *gruppoide d'inerzia* associato a G è

$$\mathrm{I}(G) := \begin{cases} (x,\varphi) \in \mathrm{Obj}\,(\mathrm{I}(G)) & \Leftrightarrow \quad x \in \mathrm{Obj}(G), \varphi \in \mathrm{Aut}(x), \\ \sigma\colon x \to y \in \mathrm{Mor}\,((x,\varphi),(y,\psi)) & \Leftrightarrow \quad \begin{array}{ccc} x & \xrightarrow{\varphi} & x \\ {\scriptstyle\sigma}\downarrow & \circlearrowright & \downarrow{\scriptstyle\sigma} \\ y & \xrightarrow[\psi]{} & y. \end{array} \end{cases}$$

inoltre si definisce una proiezione $\pi\colon \mathrm{I}(G) \to G$ ponendo $\pi(x,\varphi) := x$ e $\pi(\sigma) := \sigma$.

2.30 exercise. Esiste un naturale 2-morfismo η che rende il diagramma

$$\begin{tikzcd} \mathrm{I}(G) \arrow[r, "\pi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\overset{\eta}{\Nearrow}"] & G \arrow[d, "\Delta_G"] \\ G \arrow[r, "\Delta_G"'] & G \times G \end{tikzcd}$$

2-cartesiano.

Solution. We have to define $\eta_{(x,\varphi)}\colon (x,x) \to (x,x)$, that is, $\eta_{(x,\varphi)} = (\alpha_{(x,\varphi)}, \beta_{(x,\varphi)})$. Moreover, both α and β have to commute with any morphism $\sigma\colon x \to y$. The natural answer is to set $\alpha_{(x,\varphi)} = \varphi = \beta_{(x,\varphi)}$, since φ commutes with σ by definition of $\mathrm{I}(G)$. $\square$

3 Schemes as functors

Lecture 3 (2 hours)
January 15th, 2009

As we said, after defining groupoids and seeing some of their properties, we need to reshape the definition of scheme in order to make evident the fact that their morphisms form a set. The key to do this is Yoneda lemma.

3.1 definition. If $\mathfrak{C}$ is any category, there is a natural functor

$$\mathrm{h}\colon \mathfrak{C} \to \mathrm{Fun}(\mathfrak{C}^{\mathrm{opp}}, \mathfrak{Sets})$$

associating to an object X the functor h_X defined by

$$\mathrm{h}_X(Y) := \mathrm{Mor}_{\mathfrak{C}}(Y,X), \qquad \mathrm{h}_X(f\colon Y \to Z)\colon \mathrm{Mor}_{\mathfrak{C}}(Z,X) \xrightarrow{(\bullet \circ f)} \mathrm{Mor}_{\mathfrak{C}}(Y,X).$$

3.2 lemma (Yoneda). *The natural map* $\mathrm{Mor}(\mathrm{h}_X, F) \to F(X)$ *associating* $\alpha(\mathrm{id}_X)$ *to* $\alpha\colon \mathrm{h}_X \Rightarrow F$ *is a bijection for every* $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$.

3.3 corollary. *The functor* h *is fully faithful.*

Proof. By Yoneda lemma, $\mathrm{Mor}(\mathrm{h}_X, \mathrm{h}_Y)$ is bijective to $\mathrm{h}_Y(X) := \mathrm{Mor}(X,Y)$. $\square$

3.4 definition. A functor $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$ is called *representable* if there exist $X \in \mathrm{Obj}(\mathfrak{C})$ and $\alpha \in F(X)$ (that is, $\alpha\colon \mathrm{h}_X \Rightarrow F$) such that α is a natural equivalence. In such a situation, we say that X (or better, the couple (X, α)) *represents* F.

We can view the property that fibered product is defined up to canonical isomorphism in another way thanks to representability.

3.5 lemma. *If both* (X, α) *and* (X', α') *represent* F, *then there is a unique isomorphism* $f: X \to X'$ *such that the diagram*

$$\begin{tikzcd} \mathrm{h}_X \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & \mathrm{h}_{X'} \arrow[dl, "\alpha'"] \\ & F & \end{tikzcd}$$

commutes.

In particular, there could be many isomorphisms, but only one commutes with α and α'.

3.6 definition. Let $\mathfrak{C}$ be a category; we say that *fiber products exist* in $\mathfrak{C}$ if we can complete every diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

to a cartesian diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

3.7 lemma. *For any category* $\mathfrak{C}$, *the category* $\mathrm{Fun}(\mathfrak{C}^{\mathrm{opp}}, \mathfrak{Sets})$ *has fiber products.*

Proof. Given such a diagram, let $W(A) := X(A) \times_{Z(A)} Y(A)$; for a morphism $f: A \to B$, we have projections $W(B) \to X(B)$ and $W(B) \to Y(B)$ and also the maps $X(f): X(B) \to X(A)$ and $Y(f): Y(B) \to Y(A)$; since $W(B)$ is a fiber product, we have an induced map $W(f): W(B) \to W(A)$. It is an easy check that W is a functor and that there are natural morphisms $W \to X$ and $W \to Y$ such that W with these morphisms complete a cartesian diagram. $\square$

3.8 exercise. A category $\mathfrak{C}$ has fiber product if and only if for every $f: X \to Z$ and $g: Y \to Z$, the functor $\mathrm{h}_X \times_{\mathrm{h}_Z} \mathrm{h}_Y$ is representable.

In some sense, we proved that if we enlarge enough the category (for example considering the larger category of contravariant functors to $\mathfrak{Sets}$) we can always assume that we work in a category with fiber products.

Thanks to Yoneda lemma, we know that we can think a scheme as a contravariant functor from schemes to sets. This seems a bit circular, so we try to replace this with some precedent object. In particular, we assume to know affine schemes and we consider for a generic scheme X the restriction of h_X to affine schemes:

$$\mathrm{h}_X \colon \mathfrak{AffSch}^{\mathrm{opp}} \to \mathfrak{Sets}\,.$$

3.9 REMARK. As in Yoneda lemma, we get functors

$$\mathfrak{Sch} \to \mathrm{Fun}(\mathfrak{AffSch}^{\mathrm{opp}} \to \mathfrak{Sets}),$$

but this could be no longer an equivalence.

3.10 THEOREM.

1. *This functor is a fully faithful;*

2. *we can characterize its essential image, that is, we can describe which functors are isomorphic to* h_X *for some scheme* X*, without using the definition of schemes.*

3.11 COROLLARY. *Schemes forms a full subcategory of* $\mathrm{Fun}(\mathfrak{AffSch}^{\mathrm{opp}}, \mathfrak{Sets})$.

So we can define scheme in a complete different way from the usual one, assuming only the knowledge of affine schemes (or equivalently finitely generated K-algebras). A way to understand this, is to think that a functor $\mathfrak{AffSch}^{\mathrm{opp}} \to \mathfrak{Sets}$ encodes all possible charts from some affine schemes to our yet to be defined scheme, instead of just selecting some of them as we do in the usual way.

Proof of Theorem 3.10, part 1. Let X and Y be schemes so that $\mathrm{h}_X, \mathrm{h}_Y \colon \mathfrak{AffSch}^{\mathrm{opp}} \to \mathfrak{Sets}$. We have to prove that $\mathrm{Mor}(X, Y) \to \mathrm{Mor}(\mathrm{h}_X, \mathrm{h}_Y)$ is a bijection; to do so we provide an inverse.

Assume X is separated; in particular, for every $U, V \subseteq X$ open affine, also $U \cap V$ is open and affine; choose an open affine cover $\{U_i \mid i \in I\}$ of X via inclusions $\alpha_i \colon U_i \to X$. Let $f \colon \mathrm{h}_X \to \mathrm{h}_Y$; then $f(\alpha_i) \in \mathrm{h}_Y(U_i)$, that is $f(\alpha_i) \colon U_i \to Y$. As usual define $U_{i,j} := U_i \cap U_j$, with injections $s_{i,j} \colon U_{i,j} \to U_i$ and $t_{i,j} \colon U_{i,j} \to U_j$. Then

$$\begin{tikzcd}
& U_i \arrow[dr, "\alpha_i"] & \\
U_{i,j} \arrow[ur, "s_{i,j}"] \arrow[rr, "\alpha_{i,j}"] \arrow[dr, "t_{i,j}"'] & & X \\
& U_j \arrow[ur, "\alpha_j"'] &
\end{tikzcd}$$

commutes and we get maps $s^\star_{i,j} \colon \mathrm{h}_X(U_i) \to \mathrm{h}_X(U_{i,j})$ and $t^\star_{i,j} \colon \mathrm{h}_X(U_j) \to$

$\mathrm{h}_X(U_{i,j})$ that, applying f, we have

$$f(\alpha_i)|_{U_{i,j}} = f(\alpha_i) \circ s^{\star}_{i,j} = f(\alpha_{i,j}) = f_(\alpha_j) \circ t^{\star}_{i,j} = f(\alpha_j)|_{U_{i,j}}.$$

Therefore we can define $g\colon X \to Y$ in such a way that $g|_{U_i} = f(\alpha_i)$ and g is a uniquely determined morphism of schemes. It is now easy to check that this is really the inverse.

In the nonseparated case, the argument is the same but we have to cover again the intersections $U_{i,j}$ by affine open sets and the proof is just notationally more complicated. □

Then taking either all schemes or only affine schemes, we get two equivalent categories $\mathrm{Fun}(\mathfrak{Sch}^{\mathrm{opp}}, \mathfrak{Sets})$ and $\mathrm{Fun}(\mathfrak{AffSch}^{\mathrm{opp}}, \mathfrak{Sets})$. To prove the second part of Theorem 3.10, we need a criterion which tell us when a functor is in the essential image of h. The key concept is the next proposition, which will be proved later.

3.12 proposition. *Let X be a scheme; then the functor h_X is a sheaf in the Zariski topology on $\mathfrak{AffSch}$ in the following sense: if S is an affine schemes and $\{U_i \mid i \in I\}$ is an affine open cover of S (in particular, $U_{i,j}$ are also open affine), then the sequence*

$$\bullet \longrightarrow \mathrm{h}_X(S) \xrightarrow{\mathrm{h}_X(\alpha_i)} \prod \mathrm{h}_X(U_i) \underset{\mathrm{h}_X(t_{i,j})}{\overset{\mathrm{h}_X(s_{i,j})}{\rightrightarrows}} \prod \mathrm{h}_X(U_{i,j})$$

is exact. This means that $\mathrm{h}_X(\alpha_i)$ is injective and its image is the locus where the two other maps coincide. In other words, given for every $i \in I$ morphisms $f_i \in \mathrm{h}_X(U_i)$, then exists $f \in \mathrm{h}_X(S)$ such that $f|_{U_i} = f_i$ (or $\mathrm{h}_X(\alpha_i)(f) = f_i$) if and only if $f_i|_{U_{i,j}} = f_j|_{U_{i,j}}$ (or $\mathrm{h}_X(s_{i,j})(f_i) = \mathrm{h}_X(t_{i,j})(f_j)$); if so, then f is unique.

Another important and more difficult theorem is that h_X is a sheaf also in the étale topology.

So far we have shown that the category of schemes is equivant to a full subcategory of the category of sheaves of sets on $\mathfrak{AffSch}$ for the Zariski (and we said it is true also for étale) topology. In particular we found what we were searching: now it is evident that morphisms of schemes forms a set, since the sheaves are of sets. How do we define sheaves of groupoids? The notion of presheaf of sets is just the one of contravariant functor to $\mathfrak{Sets}$; then a presheaf of groupoids is just a contravariant functor to $\mathfrak{Groupoids}$, adapted to the fact that groupoids form a 2-category. After that we have to add glueing conditions to make a sheaf, but we will see these later.

One could ask if this is the best path to define stacks. Couldn't there be a definition that starts from a topological spaces like the objects we are acquainted to? Sure it is possible to associate a topological space to every stack; in particular orbifolds (that are, complex or symplectic manifolds that are equivalent to Deligne-Mumford stacks) are defined starting from a topological space. Indeed the definition of orbifold is much easier but it has an important problem.

3.13 PROBLEM. With the orbifold approach, the objects are very easy to define, the morphisms are messy and the 2-morphisms are yet not defined.

So let's go back to our path to the definition of presheaves of groupoids.

3.14 DEFINITION. Let $\mathfrak{C}$ be a category; a *pseudofunctor* from $\mathfrak{C}$ to $\mathfrak{Groupoids}$ is denoted as $F\colon \mathfrak{C} \to \mathfrak{Groupoids}$ and is the data of:

- for every $X \in \mathrm{Obj}(\mathfrak{C})$, a groupoid $F(X)$;
- for every $f\colon X \to Y$, a functor $F(f)\colon F(Y) \to F(X)$.
- for every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$, a 2-commutative diagram

$$
\begin{tikzcd}
F(X) & & F(Z) \arrow[ll, "F(g\circ f)"'] \arrow[dl, "F(g)"] \\
& F(Y) \arrow[ul, "F(f)"] \arrow[u, phantom, "\Uparrow\alpha_{f,g}"] &
\end{tikzcd}
$$

The data is subject to these conditions:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$;
- when either f or g is the identity, then $\alpha_{f,g}$ is the identity;
- for every sequence $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, a 2-commutative tetrahedron (seen from above and from the bottom)

$$
\begin{tikzcd}
F(X) \arrow[rr, phantom, "\Leftarrow\alpha_{g\circ f,h}"] & & F(Z) \arrow[ll, bend right, "F(g\circ f)"'] \arrow[ddl, bend left, "F(g)"] \\
& F(W) \arrow[ul, "F(h\circ g\circ f)"] \arrow[ur, "F(h)"'] \arrow[d, "F(h\circ g)"] \arrow[ur, phantom, shift right=4ex, "\Swarrow\alpha_{h,g}"] \arrow[dl, phantom, "\Nearrow\alpha_{f,h\circ g}"] & \\
& F(Y) \arrow[uul, bend left, "F(f)"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(Z) \arrow[rr, bend left, "F(g\circ f)"] \arrow[dr, bend right, "F(g)"'] & & F(X) \\
& F(Y) \arrow[ur, bend right, "F(f)"'] \arrow[u, phantom, "\Uparrow\alpha_{f,g}"] &
\end{tikzcd}
$$

  and we ask that the two 2-commutative diagrams commutes.

The definition is the natural extension of the one of functors, once changed equalities of functors to 2-isomorphisms. Actually, the first two condition do not follow this convention, but one sees that keeping this equalities does not rule out an important part of functors.

We encounter this approach every time we mess with groupoids and 2-category: we change equality to 2-isomorphisms, and request compatibility on the superior level. If for functors the data are something for every object and every arrow, subject to a condition on a sequence of two arrows, for pseudofunctors the data are something for every object, every arrow and every sequence of two arrows, subject to a condition for every sequence of three arrows.

3.15 example. Consider the mapping $V\colon \mathfrak{Manifolds} \to \mathfrak{Groupoids}$ where:

- $V(M)$ is the groupoids with objects rank r vector bundles on M and morphisms are isomorphisms of vector bundles;
- for every $f\colon M \to N$, a $\mathfrak{C}^\infty$ map that associated to a rank r vector bundle $E \to N$ the bundle $f^\star E \to M$ where $f^\star E := M \times_N E$;
- for a sequence $P \xrightarrow{g} M \xrightarrow{f} N$, a 2-morphism $\alpha_{f,g}$ defined by the canonical isomorphisms $\alpha_{f,g}(E)\colon g^\star f^\star E \to (f \circ g)^\star E$.

TODO

Notice that already does not hold anymore $V(\mathrm{id}_E) = \mathrm{id}_{V(E)}$. Indeed, $\mathrm{id}^\star E$ inside $E \times N$ is the graph of $\pi\colon E \to N$. In the same way, given $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ and $\pi\colon E \to N$, then $(g \circ f)^\star E \subseteq M_1 \times E$ but $f^\star g^\star E \subseteq M_1 \times M_2 \times E$: they are different, altough canonically isomorphic as requested. Notice also that we never used the fact that $E \to N$ is a vector bundle: e.g. we could have done the same procedure with submersion of scheme of relative dimension d there π is smooth, or proper, or projective.

3.16 remark. Recall we defined maps $\pi_0\colon \mathfrak{Groupoids} \to \mathfrak{Sets}$ and $\mathfrak{Sets} \to \mathfrak{Groupoids}$ that considers a set as a groupoids with no nontrivial isomorphisms. There are associated maps

$$\mathrm{PsFun}(\mathfrak{C}^{\mathrm{opp}}, \mathfrak{Groupoids}) \leftrightarrow \mathrm{Fun}(\mathfrak{C}^{\mathrm{opp}}, \mathfrak{Sets}).$$

Proof. If F is a pseudofunctor, then we define $\pi_0(F)(X) := \pi_0(F(X))$ and $\pi_0(F)(f) := \pi_0(f)$ and this is a functor $\mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$, since via the 2-morphisms $\alpha(f,g)$, all that should be equal in the right side is isomorphic in the left side, so is equal once cramped by π_0.

Conversely, given $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$, we can extend it to a pseudofunctor $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$: this is a very particular pseudofunctor since all 2-morphisms are identities. □

3.17 corollary. *We can associate to every scheme a pseudofunctors*

$$\mathfrak{AffSch}^{\mathrm{opp}} \to \mathfrak{Groupoids}\,.$$

So far we have defined pseudofunctor and associate a pseudofunctor to a scheme as before we associate a functor to a scheme. Before we could see all the category $\mathfrak{Sch}$ as a full subcategory of the category of functors, so now we have to define the category of pseudofunctors, that will be actually a 2-category.

3.18 definition. If F, G are pseudofunctors, a *morphism of pseudofunctor* $a\colon F \to G$ will be:

- for every $X \in \mathrm{Obj}(\mathfrak{C})$, a functor $a_X \colon F(X) \to G(X)$;
- for every $f\colon X \to Y$ a 2-commuting diagram

$$\begin{tikzcd} F(X) \arrow[r, "a_X"] & G(X) \\ F(Y) \arrow[u, "F(f)"] \arrow[r, "a_Y"'] \arrow[ur, phantom, "\Searrow"] & G(Y) \arrow[u, "G(f)"'] \end{tikzcd}$$

  such that

$$\begin{tikzcd} F(X) \arrow[rr] & & G(X) & \\ & F(Y) \arrow[ul] \arrow[rr] & & G(Y) \arrow[ul] \\ F(Z) \arrow[uu] \arrow[ur] \arrow[rr] & & G(Z) \arrow[uu] \arrow[ur] & \end{tikzcd}$$

  is 2-commutative, where the 2-morphism on the triangular faces are the ones defined by the pseudofunctors F and G, while the 2-morphisms on the rectangular faces are the one defined before.

3.19 DEFINITION. Given $a, b\colon F \to G$ where F and G are pseudofunctors, then a 2-*morphism of pseudofunctors* between a and b is denoted by $\alpha\colon a \Rightarrow b$ and is the datum of:

- for every $X \in \mathrm{Obj}(\mathfrak{C})$, a 2-morphism $\alpha_X\colon a_X \Rightarrow b_X$ such that for every $f\colon X \to Y$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, bend left, "a_X"] \arrow[r, bend right, "b_X"'] & G(X) \\ F(Y) \arrow[u, "F(f)"] \arrow[r, bend left, "a_Y"] \arrow[r, bend right, "b_Y"'] & G(Y) \arrow[u, "G(f)"'] \end{tikzcd}$$

  is a 2-commutative cylinder.

Lecture 4 (2 hours)
January 20^{th}, 2009

Recall that our aim is to embed $\mathfrak{Sch}$ into a larger category of algebraic stacks, that we want to be a 2-category. In particular, if X is an algebraic stack, whatever it is supposed to be, we would like to associate to X a functor $\mathrm{h}_X\colon \mathfrak{Sch} \to \mathfrak{Groupoids}$ such that $\mathrm{h}_X(Y)$ contains all morphisms from Y to X regarded as algebraic stacks. Indeed, what we do is to define X using h_X, that turns to be a pseudofunctor.

The main difficulty in the definition of pseudofunctor is the choice of pullbacks. Given a morphism of stack $f\colon X \to Y$ and a vector bundle $E \to Y$, we can define the strict pullback as $f^\star E := X \times_Y E$ as a set with added structure; but this works only for algebraic variety and not for generic schemes. Indeed is not natural to define one pullback, but to define pullbacks up to a canonical morphism. This is already useful without algebraic stacks; for example in the definition of fiber product of scheme and of pullbacks of quasi-coherent sheaves.

4 Categories fibered in groupoids

In the following we will see another approach to algebraic stacks, namely *category fibered in groupoids* (we will say *CFG*) and in particular how to switch from one approach to the other and why they are both important.

Let $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$ be a functor; we want to associate to F a category $\mathfrak{C}_F$ and a functor $\mathfrak{C}_F \to \mathfrak{C}$. After that we will see how to associate the same data to a functor $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$.

4.1 definition. For a morphism $\alpha\colon S \to S'$ in $\mathfrak{C}$, let $\alpha^\star := F(\alpha)\colon F(S') \to F(S)$. With this notation, we define

$$\mathfrak{C}_F := \begin{cases} (S,\xi) \in \mathrm{Obj}\,(\mathfrak{C}_F) & \Leftrightarrow \quad S \in \mathrm{Obj}(\mathfrak{C}), \xi \in F(S), \\ \alpha \in \mathrm{Mor}\,((S,\xi),(S',\xi')) & \Leftrightarrow \quad \alpha\colon S \to S' \text{ such that } \xi = \alpha^\star \xi'. \end{cases}$$

4.2 exercise. Check that $\mathfrak{C}_F$ is a category and that $\pi\colon \mathfrak{C}_F \to \mathfrak{C}$ defined by $\pi(S,\xi) := S$ and $\pi(\alpha) := \alpha$ is a covariant functor.

4.3 definition. Let $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ be a covariant functor and $S \in \mathrm{Obj}(\mathfrak{C})$; the *fiber* of $\mathfrak{C}'$ over S is the subcategory of $\mathfrak{C}$

$$\mathfrak{C}'_S := \begin{cases} S' \in \mathrm{Obj}\left(\mathfrak{C}'_S\right) & \Leftrightarrow \quad S' \in \mathrm{Obj}(\mathfrak{C}'), \pi(S') = S, \\ \alpha \in \mathrm{Mor}\,(S',T') & \Leftrightarrow \quad \pi(S') = S = \pi(T'), \pi(\alpha) = \mathrm{id}_S\,. \end{cases}$$

4.4 lemma. *Let $\pi\colon \mathfrak{C}_F \to \mathfrak{C}$ the category associated to a contravariant functor $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$, then each fiber of π is a set.*

Proof. The objects of $\mathfrak{C}_{F,S}$ are $\{(S,\xi) \mid \xi \in F(S)\}$, while its morphisms are morphism $\alpha\colon S \to S$ mapped to the identity and such that $\alpha^\star\xi' = \xi$; in particular, $\alpha = \mathrm{id}_S$ and $\xi' = \xi$. □

This lemma explains why we take only morphisms mapping to the identity and not the full subcategory. In general, if we took the full subcategory, the fiber would not be a set because we could have more morphisms.

4.5 remark. Not every $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ with sets as fiber comes from a functor; for example, take a set $\mathfrak{C}'$ as a category, fix an object $S \in \mathrm{Obj}(\mathfrak{C})$ and define π to be the constant functor $\pi(X) := S$ and $\pi(f) = \mathrm{id}_S$. This is a functor with only one fiber and this fiber is a set, but if there are other morphisms $S \to S$, π cannot come from a functor F since there are no non-trivial pullbacks.

We want to restate in this language the fact that pullbacks are unique up to a canonical isomorphism.

4.6 DEFINITION. Fix $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ a covariant functor; a π-*commutative* diagram will be

$$\begin{tikzcd} \xi_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \xi_2 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

where $\tilde{\varphi}\colon \xi_1 \to \xi_2$ is a morphism in $\mathfrak{C}'$ and $\pi(\tilde{\varphi}) = \varphi$. A π-commutative diagram is π-*cartesian* if for every π-commutative diagram

$$\begin{tikzcd} \xi_0 \arrow[d, "\pi"] \arrow[rr, bend left, "\tilde{\alpha}"] & \xi_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \xi_2 \arrow[d, "\pi"] \\ S_0 \arrow[r, "\psi"'] \arrow[rr, bend right, "\alpha:=\varphi\circ\psi"'] & S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

that is, $\pi(\tilde{\alpha}) = \alpha$ and $\pi(\tilde{\varphi}) = \varphi$, there is a unique $\tilde{\psi}\colon \xi_0 \to \xi_1$ making the diagram 2-commutative, that is $\pi(\tilde{\psi}) = \psi$ and $\tilde{\alpha} = \tilde{\varphi} \circ \tilde{\psi}$.

In a π-commutative diagram, the upper floor lives in a category, while the lower in another category, but all maps and objects in the upper floor are mapped by π in the right maps and objects in the lower floor.

4.7 DEFINITION. A functor $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ makes $\mathfrak{C}'$ into a *fiber category* over $\mathfrak{C}$ if for every π-commutative diagram

$$\begin{tikzcd} & \xi_2 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

there exists a (not necessarily unique) π-cartesian diagram

$$\begin{tikzcd} \xi_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \xi_2 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

extending it. We say that ξ_1 is the *pullback* of ξ_2 over φ.

4.8 exercise. Let $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ be a covariant functor. Then there exists a $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$ and an isomorphism (not only an equivalence) $\mathfrak{C}_F \to \mathfrak{C}'$ if and only if

1. $\mathfrak{C}' \to \mathfrak{C}$ is a fibered category and
2. every fiber of $\mathfrak{C}'$ is a set.

Hint: in this case, π-cartesian diagram exist and are also unique.

4.9 proposition. *Let $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ be a covariant functor and assume the diagram*

$$\begin{tikzcd} \xi_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \xi_2 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

to be π-cartesian; then there is a bijection

$$\left\{ \tilde{\psi}\colon \xi_1' \to \xi_2 \;\middle|\; \begin{tikzcd} \xi_1' \arrow[r, "\tilde{\psi}"] \arrow[d, "\pi"] & \xi_2 \arrow[d, "\pi"'] \\ S_1 \arrow[r, "\varphi"] & S_2 \end{tikzcd} \text{ is } \pi\text{-cartesian} \right\} \longleftrightarrow$$

$$\longleftrightarrow \left\{ \tilde{\alpha}\colon \xi_1' \to \xi_1 \;\middle|\; \begin{tikzcd} \xi_1' \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi"] & \xi_1 \arrow[d, "\pi"'] \\ S_1 \arrow[r, "\mathrm{id}_{S_1}"] & S_1 \end{tikzcd} \text{ is } \pi\text{-cartesian} \right\}$$

Proof. Given $\tilde{\alpha}$, we define $\tilde{\psi} := \tilde{\varphi} \circ \tilde{\alpha}$; we get $\pi(\tilde{\psi}) = \varphi \circ \mathrm{id}_S = \varphi$ so the diagram is π-commutative and since we have two adjacent π-cartesian diagram, the one constructed by merging the two is again π-cartesian. Conversely, consider the diagram

$$\begin{tikzcd} \xi_1' \arrow[rr, bend left, "\tilde{\psi}"] \arrow[d, "\pi"] & \xi_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \xi_2 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\mathrm{id}_{S_1}"'] & S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd}$$

and by definition of π-cartesian, there exists a unique $\tilde{\alpha}\colon \xi_1' \to \xi_1$ completing the π-commutative diagram and to check that the left square is π-cartesian we can reverse te role of $\tilde{\psi}$ and $\tilde{\varphi}$ and observe that whenever we have a diagram

$$\begin{tikzcd} \xi_1' \arrow[r, "\tilde{\alpha}"] \arrow[d, "\pi"'] & \xi_1 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\mathrm{id}_{S_1}"'] & S_1 \end{tikzcd}$$

then $\tilde{\alpha}$ is an isomorphism if and only if the diagram is π-cartesian. □

In particular, to study the problem of how non-unique the pullback is, we have to study the arrows lifting the identity of S_1. Now we are ready to specialize to the notion of category fibered in groupoids.

4.10 DEFINITION. A *category fibered in groupoids* (or *CFG*) over a category $\mathfrak{C}$ is a category $\mathfrak{C}'$ with a functor $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ such that

1. π makes $\mathfrak{C}'$ into a fibered category over $\mathfrak{C}$ and
2. every fiber is a groupoid.

4.11 EXERCISE. Show that if $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ is a CFG, then every π-commutative diagram is π-cartesian.

What we gain is that defining CFGs is much more easy than defining pseudofunctors.

4.12 EXERCISE. Let $\mathfrak{C} := \mathfrak{Sch}$ (or $\mathfrak{Var}$).

1. Consider the category

$$\mathfrak{Qcoh} := \begin{cases} (X, \mathscr{E}) \in \operatorname{Obj}(\mathfrak{Qcoh}) & \Leftrightarrow \quad X \in \mathfrak{C}, \mathscr{E} \in \mathfrak{Qcoh}(X), \\ (f, \varphi) \in \operatorname{Mor}((X, \mathscr{E}), (X', \mathscr{E}')) & \Leftrightarrow \quad f\colon X \to X', \varphi\colon \mathscr{E} \xrightarrow{\sim} f^\star \mathscr{E}'. \end{cases}$$

   Here φ is defined up to canonical isomorphism (of the pullback). Let $\pi\colon \mathfrak{Qcoh} \to \mathfrak{C}$ be a functor defined by $\pi(X, \mathscr{E}) := X$ and $\pi(f, \varphi) := f$; then π is a CFG and the same is true for the full subcategory of coherent, or locally free, or locally free of rank r vector bundles.

2. Let

$$\mathfrak{Flat} := \begin{cases} \pi_1 \in \operatorname{Obj}(\mathfrak{Flat}) & \Leftrightarrow \quad \pi_1\colon X_1 \to S_1 \text{ is a flat family,} \\ (\tilde{\varphi}, \varphi) \in \operatorname{Mor}(\pi_1, \pi_2) & \Leftrightarrow \quad \begin{tikzcd} X_1 \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ S_1 \arrow[r, "\varphi"'] & S_2 \end{tikzcd} \text{ is cartesian.} \end{cases}$$

   Show that $\pi\colon \mathfrak{Flat} \to \mathfrak{Sch}$ defined by $\pi(\pi_1) := S_1$ and $\pi(\tilde{\varphi}, \varphi) := \varphi$ makes $\mathfrak{Flat}$ into a CFG over $\mathfrak{C}$. The same for the full subcategories of proper, or projective, or smooth morphism; we can also add assumptions on the fibers of π_1 and obtaining again a CFG.

4.13 proposition. *Let $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ be a pseudofunctor; then we define*

$$\mathfrak{C}_F := \begin{cases} (S,\xi) \in \mathrm{Obj}\,(\mathfrak{C}_F) & \Leftrightarrow \quad S \in \mathrm{Obj}(\mathfrak{C}), \xi \in \mathrm{Obj}(F(S)), \\ (\varphi,\tilde{\varphi}) \in \mathrm{Mor}\,((S,\xi),(S',\xi')) & \Leftrightarrow \quad \varphi\colon S \to S', \tilde{\varphi}\colon \xi \to \varphi^\star\xi'. \end{cases}$$

Define the composition of $(\psi,\tilde{\psi})\colon (S_2,\xi_2) \to (S_3,\xi_3)$ and $(\varphi,\tilde{\varphi})\colon (S_1,\xi_1) \to (S_2,\xi_2)$ to be the map $(\alpha,\tilde{\alpha})\colon (S_1,\xi_1) \to (S_3,\xi_3)$ with

$$\alpha := \psi \circ \varphi, \qquad\qquad \tilde{\alpha}\colon \xi_1 \to \alpha^\star\xi_3 = (\psi\circ\varphi)^\star\xi_3;$$

notice that the target of $\tilde{\alpha}$ is only canonically isomorphic to $\varphi^\star \circ \psi^\star\xi_3$ via $\eta(\psi,\varphi)\colon \varphi^\star \circ \psi^\star\xi_3 \to (\psi\circ\varphi)^\star\xi_3$; so we define $\tilde{\alpha} := \eta(\psi,\varphi)\circ\varphi^\star(\tilde{\psi})\circ\tilde{\varphi}$. Then

1. *$\mathfrak{C}_F$ is a category;*
2. *$\pi\colon \mathfrak{C}_F \to \mathfrak{C}$ defined by $\pi(S,\xi) := S$ and $\pi(\varphi,\tilde{\varphi}) := \varphi$ makes $\mathfrak{C}_F$ into a CFG over $\mathfrak{C}$.*

4.14 exercise. Prove the proposition and observe that pseudofunctor axioms assure that $\mathfrak{C}_F$ is a category (in particular, that composition is associative).

4.15 theorem. *Let $\pi\colon \mathfrak{C}' \to \mathfrak{C}$ be a CFG; then there exists a pseudofunctor $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ and an equivalence $\alpha\colon \mathfrak{C}' \to \mathfrak{C}_F$ inducing π. Moreover the couple (F,α) is unique up to unique equivalence of groupoids.*

Sketch of the proof. Define for $S \in \mathrm{Obj}(\mathfrak{C})$, $F(S) := \mathfrak{C}'_S$; we have to define pullbacks, that is, given a π-commutative diagram

$$\begin{tikzcd} & \xi_2 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\varphi"'] & S_2, \end{tikzcd}$$

to prove that the set of $\tilde{\varphi}\colon \xi_1 \to \xi_2$ that makes the diagram π-cartesian is non-empty and select one for each diagram with the axiom of choice. In other words, for any morphism $\varphi\colon S_1 \to S_2$ and each lifting ξ_1 of S_2, we choose a morphism $\varphi\colon \xi_1 \to \xi_2$ making the diagram π-cartesian. We may assume (but is not really necessary) that for $\varphi = \mathrm{id}_S$ and ξ over S we choose id_ξ. Now, if $\alpha\colon \xi_2 \to \xi_2'$ is a morphism in $F(S)$ (in particular, it is over id_{S_2}) then we have to define $\varphi^\star\alpha = F(\varphi)(\alpha)$. We write the commutative diagram

$$\begin{tikzcd} \varphi^\star\xi_2' \arrow[r, dashed, "\beta"'] \arrow[d, "\pi"] \arrow[rrr, bend left, "\tilde{\varphi}_{\xi_2'}"] & \varphi^\star\xi_2 \arrow[r, "\tilde{\varphi}_{\xi_2}"'] \arrow[d, "\pi"] & \xi_2 \arrow[r, "\alpha"'] \arrow[d, "\pi"] & \xi_2' \arrow[d, "\pi"] \\ S_1 \arrow[r, "\mathrm{id}_{S_1}"] \arrow[rrr, bend right, "\varphi"'] & S_1 \arrow[r, "\varphi"] & S_2 \arrow[r, "\mathrm{id}_{S_2}"] & S_2 \end{tikzcd}$$

from which we know that there exists a unique β by π-cartesianity; now, the center and the right squares are π-cartesian and so is the left one; therefore β is an isomorphism and we define $\varphi^\star\alpha := \beta^{-1}$. □

4.16 EXERCISE. Conclude the definition of F, namely given $S_0 \xrightarrow{\varphi} S_1 \xrightarrow{\psi} S_2$, construct a 2-morphism $\eta\colon (\psi\circ\varphi)^\star \Rightarrow \varphi^\star \circ \psi^\star$.

Up to now, we have the category of CFGs over $\mathfrak{C}$ (we will see promptly how this is naturally a 2-category) and a 2-category of pseudofunctors $\mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ and a map $F \mapsto \mathfrak{C}_F$. We want to prove that this is an equivalence of 2-categories (even if we will not fill all details). In the literature, all main references define stacks from CFGs, but with a different graphical aspect; namely, they say that we have lifting of morphisms (for every $\varphi\colon S_1 \to S_2$ and a lifting of S_2 through π, there exists a lifting of φ) and lifting of triangles (given a commutative triangle and a lifting of two arrows out of three, we have a unique lifting of the third). The reason why most of the references do not even mention pseudofunctors is that as we saw, many geometric objects have a natural description as CFGs, but to obtain a pseudofunctor we have to use the axiom of choice.

If $\tilde{\mathfrak{C}}$ is a 2-category and $\mathfrak{C} \subseteq \tilde{\mathfrak{C}}$ is a full subcategory (that is, it has all 2-morphisms of $\tilde{\mathfrak{C}}$), then to every object $X \in \mathrm{Obj}(\tilde{\mathfrak{C}})$ we can associate a pseudofunctor $\mathrm{h}_X\colon \mathfrak{C} \to \mathfrak{Groupoids}$, that in the case of stacks gives a canonical description of a CFG into a pseudofunctor. But in general case there is not a canonical descriptions of a CFG as a pseudofunctor. *TODO*

4.17 DEFINITION. We make CFGs over $\mathfrak{C}$ into a 2-category as follows:

$$\mathfrak{CFG} := \begin{cases} (\mathfrak{C}', \pi) \in \mathrm{Obj}(\mathfrak{CFG}) & \Leftrightarrow \quad \pi\colon \mathfrak{C}' \to \mathfrak{C} \text{ is a CFG,} \\ F \in \mathrm{Mor}((\mathfrak{C}',\pi),(\mathfrak{C}'',\pi')) & \Leftrightarrow \quad \begin{array}{c} F\colon \mathfrak{C}' \to \mathfrak{C}'' \\ \text{commutes with } \pi \text{ and } \pi', \end{array} \\ (\eta_\xi) \in 2\text{-Mor}(F, F') & \Leftrightarrow \quad \eta_\xi\colon F(\xi) \to F'(\xi), \pi'(\eta_\xi) = \mathrm{id}_{\pi(\xi)}\,. \end{cases}$$

Lecture 5 (2 hours)
January 21st, 2009

We followed the approach of viewing schemes as a full subcategory of a category of contravariant functors $\mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$; we saw that such a functor is equivalent to a category fibered in sets, $\mathfrak{C} \to \mathfrak{Sch}$. In the same way, we want to define algebraic stacks as a full 2-subcategory of pseudofunctors $\mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Groupoids}$. We already saw how to associate a CFG to a pseudofunctor; the converse require the axiom of choice.

The main question now is: how do we recognize schemes among all contravariant functor $\mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$? We recall in the following some notions we already mention.

4.18 REMARK. If X is a scheme, then $\mathrm{h}_X\colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$ is a sheaf in the Zariski topology; this means that if we have local morphisms that glue, there is a unique glued morphism. More precisely, given a scheme S and an open cover $\{S_i\}$, then if $\varphi_i \in \mathrm{h}_X(S_i)$ are such that $\varphi_i|_{S_{i,j}} = \varphi_j|_{S_{i,j}}$, then there exists a unique $\varphi \in \mathrm{h}_X(S)$ such that $\varphi|_{S_i} = \varphi_i$.

4.19 corollary. *If X is a scheme, then* h_X *is determined by its restriction to* $\mathfrak{AffSch}$.

4.20 remark. In fact h_X is a sheaf for other Grothendieck topologies on $\mathfrak{Sch}$. Consider a collection of maps $\mathscr{S} := \{\alpha_i \colon S_i \to S\}$ with $S_{i,j} := S_i \times_S S_j$; then $\mathscr{S}$ is an open cover of S if $S = \bigcup \alpha_i(S_i)$ and

- all α_i are étale in the étale topology;
- all α_i are smooth in the smooth topology;
- all α_i are faithfully flat with finite presentation (respectively faithfully flat and quasicompact) in the fppf (respectively fpqc) topology.

4.21 remark. Let $f \colon X \to Y$ a morphism of schemes; then

$$f \text{ étale } \Rightarrow f \text{ smooth } \Rightarrow f \text{ flat } \Rightarrow f \text{ open.}$$

4.22 definition. A *stack* over a category $\mathfrak{C}$ with a fixed Grothendieck topology is a pseudofunctor $F \colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ such that

1. for every $X \in \mathrm{Obj}(\mathfrak{C})$ and for every $\xi, \xi' \in \mathrm{Obj}(F(X))$, $\mathrm{Mor}(\xi, \xi')$ is a sheaf, and
2. every descent datum is effective.

Our main example is when $\mathfrak{C} = \mathfrak{Sch}$ with the étale topology. This is the standard Artin definition given in 1974; to understand better what this means, we will explain it in greater details: as a slogan, the two conditions mean jointly that stacks glue like vector bundles. Indeed, considering vector bundles, the two conditions become the following.

1. Morphisms are a sheaf means that to define a morphism of vector bundles $\varphi \colon E_1 \to E_2$ is equivalent to give an open cover $\{U_i\}$ of the base space X and to define morphisms $\varphi_i \colon E_1|_{U_i} \to E_2|_{U_i}$ such that $\varphi_i|_{U_{i,j}} = \varphi_j|_{U_{i,j}}$.
2. Given S a scheme with an open cover $\{S_i\}$, and vector bundles $E_i \to S_i$, then to glue them to a vector bundle $E \to S$ is not enough to assume $E_i|_{S_{i,j}} \cong E_j|_{S_{i,j}}$ (in particular this does not imply E exists or that if it exists is unique, even up to isomorphism); so a *descent datum* is a collection of vector bundles $E_i \to S_i$, with isomorphisms $\varphi_{i,j} \colon E_i|_{S_{i,j}} \to E_j|_{S_{i,j}}$ such that $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ (all restricted to $S_{i,j,k}$); this descent datum is *effective* if:
    - there exists $E \to S$ with isomorphisms $\varphi_i \colon E|_{S_i} \to E_i$ such that $\varphi_{i,j} = \varphi_i \circ \varphi_j^{-1}$;
    - if (E, φ_i) and (E', φ_i') both satisfy the previous condition, then there exists a unique $\varphi \colon E \to E'$ such that $\varphi_i = \varphi_i' \circ \varphi|_{S_i}$.

Now we can rephrase the conditions in the general case.

1. If $X \in \mathrm{Obj}(\mathfrak{C})$ and $\xi, \xi' \in F(X)$, then we consider the functor

$$M \coloneqq \mathrm{Mor}(\xi, \xi') \colon (\mathfrak{C}/X)^{\mathrm{opp}} \to \mathfrak{Sets}\,;$$

If $p \colon S \to X$, then $M(p) \coloneqq \mathrm{Mor}_{F(S)}(p^\star \xi, p^\star \xi')$; given $f \colon p_1 \to p_2$, we have to define $M(f)(\alpha)$ for $\alpha \colon p_2^\star \xi \to p_2^\star \xi' \in M(p_2)$; the natural choice would be $f^\star \alpha \colon f^\star p_2^\star \xi \to f^\star p_2^\star \xi'$, but there only natural isomorphisms $\eta_\xi \colon p_1^\star \xi \to f^\star p_2^\star \xi$ and $\eta_{\xi'} \colon p_1^\star \xi' \to f^\star p_2^\star \xi'$, not equalities; so we define $M(f)(\alpha) \coloneqq \eta_{\xi'}^{-1} \circ f^\star \alpha \circ \eta_\xi$, as in the following diagram:

$$\begin{tikzcd}
f^\star p_2^\star \xi \arrow[r, "f^\star \alpha"] & f^\star p_2^\star \xi' \\
p_1^\star \xi \arrow[u, "\eta_\xi", "\wr"'] \arrow[r, dashed, "M(f)(\alpha)"'] & p_1^\star \xi' . \arrow[u, "\eta_{\xi'}"', "\wr"]
\end{tikzcd}$$

Then the first condition says that this functor M is a sheaf on $\mathfrak{C}/X$ with the induced topology from $\mathfrak{C}$.

2. A descent datum for F is:

   - an object $S \in \mathrm{Obj}(\mathfrak{C})$ with an open cover $\{\alpha_i \colon S_i \to S\}$;
   - for every i, an object $\xi_i \in F(S_i)$;
   - for every i, j, an isomorphism $\varphi_{i,j} \colon \xi_i|_{S_{i,j}} \to \xi_j|_{S_{i,j}}$ on $F(S_{i,j})$ (where $\xi_i|_{S_{i,j}} \coloneqq \pi_i^\star \xi_i$ is the natural map) such that for every i, j, k, we have a commutative diagram

$$\begin{tikzcd}
 & & p_{i,j}^\star \pi_i^\star \xi_i \arrow[dr, "p_{i,j}^\star \varphi_{i,j}"] & \\
 & p_i^\star \xi_i \arrow[ur, "\sim"'] & & p_{j,i}^\star \pi_j^\star \xi_j \arrow[d, "\wr"] \\
p_{i,k}^\star \pi_i^\star \xi_i \arrow[ur, "\wr"'] \arrow[d, "p_{i,k}^\star \varphi_{i,k}"'] & & & p_j^\star \xi_j \arrow[d, "\wr"] \\
p_{i,k}^\star \pi_k^\star \xi_k \arrow[dr, "\wr"] & & & p_{j,k}^\star \pi_j^\star \xi_j \arrow[dl, "p_{j,k}^\star \varphi_{j,k}"] \\
 & p_k^\star \xi_k \arrow[r, "\sim"] & p_{j,k}^\star \pi_k^\star \xi_k &
\end{tikzcd}$$

   in $F(S_{i,j,k})$ where the maps involved are

$$\begin{tikzcd}
S_{i,j,k} \arrow[r, "p_{i,j}"] \arrow[dr, "p_i"'] & S_{i,j} \arrow[d, "\pi_i"] \\
 & S_i
\end{tikzcd}$$

(note that in the case of vector bundles the situation was apparently simpler since we did not show the canonical isomorphisms); this datum is effective if there exists $\xi \in F(S)$ and $\varphi_i \colon \alpha_i^\star \xi \to \xi_i$ such that the diagram

$$\begin{tikzcd}
& \xi_i|_{S_{i,j}} \arrow[rr, "\varphi_{i,j}"] \arrow[dl, "\varphi_i|_{S_{i,j}}"'] & & \xi_j|_{S_{i,j}} \arrow[dr, "\varphi_j|_{S_{i,j}}"] & \\
\xi|_{S_i}|_{S_{i,j}} & & & & \xi|_{S_j}|_{S_{i,j}} \\
& & \xi|_{S_{i,j}} \arrow[ull, "\sim"] \arrow[urr, "\sim"'] & &
\end{tikzcd}$$

commutes in $F(S_{i,j})$.

4.23 exercise. The defined M is a functor.

4.24 exercise. If F is a stack, then given a descent datum $(\alpha_i \colon S_i \to S, \xi_i, \varphi_{i,j})$, and given two couples (ξ, φ_i) and (ξ', φ_i') making it effective, then there exists a unique $\alpha \colon \xi \to \xi'$ with the natural compatibility condition.

4.25 exercise. Check that the CFGs defined previously over $\mathfrak{Sch}$ are stacks.

In all these commutative diagrams, we have two kind of maps: pullbacks and natural isomorphisms between compositions of pullbacks and pullbacks of compositions. The conditions given above for pseudofunctors could probably be written in terms of CFGs, but the description will be notably longer.

4.26 fact. Let $\mathfrak{C}$ be a category with a Grothendieck topology and $\mathfrak{C}'$ be a CFG over $\mathfrak{C}$. If $F \colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ is a pseudofunctor obtained from $\mathfrak{C}'$, then being a stack is a property of the functor $\mathfrak{C}' \to \mathfrak{C}$ and not of the choice made to construct F. More precisely, $\mathrm{PsFun}(\mathfrak{C}^{\mathrm{opp}}, \mathfrak{Groupoids})$ has a natural structure of 2-category for which exists an equivalence of 2-category $\mathrm{PsFun} \to \mathfrak{CFG}$; if $\alpha \colon F \to F'$ is a morphism of pseudofunctor which is an equivalence, then F is a stack if and only if F' is.

4.27 definition. Let $F, G \colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ be pseudofunctors; then a morphism $\varphi \colon F \to G$ is the datum of

- for every $S \in \mathrm{Obj}(\mathfrak{C})$, a morphism $\varphi_S \colon F(S) \to G(S)$;

- for every morphism $f \in \mathrm{Mor}_{\mathfrak{C}}(S, T)$, a 2-morphism $\eta_f \colon \varphi_T \circ f^\star \Rightarrow f^\star \circ$

φ_S, that is, a 2-commutative diagram

$$\begin{tikzcd}
F(S) \arrow[r, "\varphi_S"] & G(S) \\
F(T) \arrow[u, "f^\star"] \arrow[r, "\varphi_T"'] \arrow[ur, phantom, "\Leftarrow\eta_f"] & G(T), \arrow[u, "f^\star"']
\end{tikzcd}$$

such that

- $\eta_{\mathrm{id}} = \mathrm{id}$;
- given $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$, the prism

$$\begin{tikzcd}
F(S_1) \arrow[rr] & & G(S_1) & \\
& F(S_2) \arrow[ul] \arrow[rr] & & G(S_2) \arrow[ul] \\
F(S_3) \arrow[uu] \arrow[ur] \arrow[rr] & & G(S_3); \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

2-commutes.

4.28 DEFINITION. If $\varphi, \psi \colon F \to G$ are morphisms of pseudofunctors, then a 2-morphism $\alpha \colon \varphi \Rightarrow \psi$ of pseudofunctors is the datum of, for every object $S \in \mathrm{Obj}(\mathfrak{C})$, a 2-morphism $\alpha_S \colon \varphi_S \Rightarrow \psi_S$ such that

$$\begin{tikzcd}
F(S) \arrow[r, bend left] \arrow[r, bend right] & G(S) \\
F(T) \arrow[u] \arrow[r, bend left] \arrow[r, bend right] & G(TY) \arrow[u]
\end{tikzcd}$$

is a 2-commutative cylinder.

4.29 REMARK. Fiber products of sets induce fiber products of functors $\mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$. So we may try to define fiber products of pseudofunctors from fiber products of groupoids.

4.30 DEFINITION. Let $F, G, H \colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ be pseudofunctors with morphisms $\varphi \colon F \to H$ and $\psi \colon G \to H$. We define $K := F \times_H G \colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ as follows:

- for every $S \in \mathrm{Obj}(\mathfrak{C})$, $K(S) := F(S) \times_{H(S)} G(S)$;
- for every $f \colon S \to T$, a morphism $K(f) \colon K(T) \to K(S)$ defined on objects by $K(f)(\xi_1, \xi_2, \alpha) := (f^\star \xi_1, f^\star \xi_2, \tilde{\alpha})$ where $\tilde{\alpha}$ is the usual composition of

$f^\star\alpha$ with the natural isomorphisms $\varphi(f^\star\xi_1) \cong f^\star\varphi(\xi_1)$ and $\varphi(f^\star\xi_2) \cong f^\star\varphi(\xi_2)$; defined on morphisms by $K(\beta_1, \beta_2) := (f^\star\beta_1, f^\star\beta_2)$.

- given $S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$, a 2-morphism $f^\star g^\star \Rightarrow (g \circ f)^\star$; but $f^\star g^\star(\xi_1, \xi_2, \alpha) = (f^\star g^\star \xi_1, f^\star g^\star \xi_2, \bullet)$ and $(g \circ f)^\star(\xi_1, \xi_2, \alpha) = ((g \circ f)^\star\xi_1, (g \circ f)^\star\xi_2, \bullet)$ so the needed 2-morphism is given by the usual natural isomorphisms.

4.31 theorem.

1. *Let $F \to H$, $G \to H$ be morphisms of pseudofunctors; then there are natural morphisms $F \times_H G \to F$ and $F \times_H G \to G$ such that the diagram*

$$\begin{tikzcd} F \times_H G \arrow[r] \arrow[d] & F \arrow[d] \\ G \arrow[r] & H \end{tikzcd}$$

   is naturally 2-cartesian.

2. *If F, G, and H are stacks, so is $F \times_H G$.*

4.32 exercise.

1. Let $\mathfrak{C}$ be a category with a Grothendieck topology, $F\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$ be a functor and $\tilde{F}\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ the induced pseudofunctor. Then F is a sheaf if and only if $\tilde{F}$ is a stack.

2. If $F, G, H\colon \mathfrak{C}^{\mathrm{opp}} \to \mathfrak{Sets}$ are sheaves, and $\varphi\colon F \to H$ and $\psi\colon G \to H$ are morphisms of functors, then $F \times_G H$ is a sheaf.

Recall that if G is a group acting on a set X, we defined the groupoid $[X/G]$ and we proved that $G \times X$ is the fiber product of X and X over $[X/G]$. We want to do the same for groupoids.

4.33 definition. Let G be a group scheme, and X be a scheme; an *action* of G on X is a morphism $a\colon G \times X \to X$ such that $a_{\{e\} \times X} = \mathrm{id}_X$ and the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "{(m, \mathrm{id}_X)}"] \arrow[d, "{(\mathrm{id}_G, a)}"'] & G \times X \arrow[d, "a"] \\ G \times X \arrow[r, "a"'] & X \end{tikzcd}$$

commutes.

4.34 definition. A principal G-bundle P over a scheme S in the étale topology is:

1. a morphism of scheme $\pi\colon P \to S$;

2. an action a of G on P such that the diagram

$$\begin{tikzcd} G \times P \arrow[r, "a"] \arrow[d, "p_2"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] & S \end{tikzcd}$$

commutes;

3. an étale cover $\{S_i \to S\}$ such that, letting $P_i := P \times_S S_i$, a induces an action of G on P_i; and there exists a section $s\colon S_i \to P_i$ such that the morphism $G \times S_i \xrightarrow{(\mathrm{id}_G, s)} G \times P_i \xrightarrow{a} P_i$ is an isomorphism.

4.35 DEFINITION. The pseudofunctor $[X/G]$ is defined by:

1. to an object S, we associate the groupoids

$$[X/G](S) := \begin{cases} (\pi, \varphi) \in \mathrm{Obj}\,([X/G](S)) & \Leftrightarrow \begin{array}{l} \pi \text{ principal } G\text{-bundle,} \\ \varphi\ G\text{-equivarant,} \end{array} \\ \alpha\colon P \to P' \in \mathrm{Mor}\,((\pi, \varphi), (\pi', \varphi')) & \Leftrightarrow \begin{array}{l} \alpha \text{ equivariant,} \\ \text{commuting with projections to } S \\ \text{and morphisms to } X. \end{array} \end{cases}$$

2. on a morphism $f\colon S \to T$, we associate the functor $f^\star$ with $f^\star(\pi, \varphi) := (\pi', \varphi')$ where $P' := P \times_T S$ with the induced G-action;

3. on a composition $S \xrightarrow{f} T \xrightarrow{g} V$, the natural map $f^\star g^\star P \to (g \circ f)^\star P$.

4.36 THEOREM.

1. *The so defined $[X/G]$ is a pseudofunctor;*

2. *it is a stack in the étale topology;*

3. *there is a natural 2-cartesian diagram*

$$\begin{tikzcd} \mathrm{h}_G \times \mathrm{h}_X \arrow[r] \arrow[d] & \mathrm{h}_X \arrow[d] \\ \mathrm{h}_X \arrow[r] & {[X/G]} \end{tikzcd}$$

4.37 NOTATION. A principal G-bundle is often called a *G-torsor*.

Lecture 6 (2 hours)
January 27th, 2009

So far, we have seen a scheme X as a functor $\mathrm{h}_X\colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$ (or, equivalently, as a category fibered in sets over $\mathfrak{Sch}$) that is also a sheaf with respect to the Zariski topology (or the étale topology) over $\mathfrak{Sch}$. We have also seen that the fiber product of two sheaves is again a sheaf with respect to the same topology.

We saw a stack as a pseudofunctor $F\colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Groupoids}$ or equivalently, as a CFG over $\mathfrak{Sch}$, that satisfy the two conditions of stacks. If $F\colon \mathfrak{Sch}^{\mathrm{opp}} \to$

$\mathfrak{Sets}$, then F satisfies these conditions if and only if it is a sheaf. We also saw that fiber product of pseudofunctors respect the property of being a stack.

How to know whether a functor $F\colon \mathfrak{Sch}^{\text{opp}} \to \mathfrak{Sets}$ is representable, that is, isomorphic to h_X for some scheme X?

- The hard answer is to use Yoneda: we have to find a scheme X and an $\alpha \in F(X) = \mathrm{Mor}(\mathrm{h}_X, F)$ such that $\alpha\colon \mathrm{h}_X \to F$ is an isomorphism.
- We have an easier necessary condition: if F is representable, then F is a sheaf in the Zariski (and also in the étale) topology.

As we will point out later, this necessary condition is not sufficient: there exist sheaves that are not isomorphic to h_X for any scheme X.

4.38 definition. A morphism $F \to G$ of functors to sets is *representable* if for every scheme S and for every $\mathrm{h}_S \to G$, the functor $F \times_G \mathrm{h}_S$ is representable.

This notion of representability of a morphism is important because we can extend to representable morphisms some properties of morphisms of schemes, as follows.

4.39 remark. If P is a property of morphism, we say that a representable morphism φ has P if and only if for every $\mathrm{h}_S \to G$ the morphism $\mathrm{h}_T \xrightarrow{\sim} F \times_G \mathrm{h}_S \to \mathrm{h}_S$ has P (that is, if and only if the morphism of schemes $T \to S$ has P). To have a well definition, we have to consider only properties P that are stable under base change.

From now on, we consider $\mathfrak{Sch}$ to be the category of schemes of finite type over an algebraically closed field K.

4.40 theorem. *Let $F\colon \mathfrak{Sch}^{\text{opp}} \to \mathfrak{Sets}$ be a sheaf; then F is representable if and only if there is a finite collection of morphisms $\alpha_i\colon \mathrm{h}_{S_i} \to F$ where for every i:*

1. *the S_i are affine schemes;*
2. *the α_i are representable;*
3. *the α_i are open embeddings;*
4. *$\alpha = \bigsqcup \alpha_i\colon \mathrm{h}_{\bigsqcup S_i} \to F$ is surjective.*

We would like to give a definition of a "representable stack" in analogy with this theorem.

There is a natural morphism

$$\mathrm{Fun}(\mathfrak{Sch}^{\text{opp}}, \mathfrak{Sets}) \to \mathrm{Fun}(\mathfrak{AffSch}^{\text{opp}}, \mathfrak{Sets}),$$

the restriction. A priori this morphism lose a lot of information, but for sheaves in the Zariski topology or in the étale topology this is an equivalence.

4.41 remark. We could use Theorem 4.40 to define schemes as sheaves on $\mathfrak{AffSch}$. The functor h_S restricted to affine schemes is often called the *functor of points*.

4.42 EXAMPLE. Fix $n > 0$ and $P \in \mathbb{Q}[t]$ a numerical polynomial (that is, $P(\mathbb{Z}) \subseteq \mathbb{Z}$); then we can define the functor $\mathrm{Hilb}^P_{\mathbb{P}^n}$ by

$$\mathrm{Hilb}^P_{\mathbb{P}^n}(S) := \left\{ Z \subseteq S \times \mathbb{P}^n \;\middle|\; \begin{array}{l} Z \text{ a closed subscheme,} \\ \text{flat over } S\text{, with each fiber} \\ \text{with Hilbert polynomial } P \end{array} \right\}.$$

Note that flatness already implies that the Hilbert polynomials of the fibers are all equal. In particular, $\mathrm{H}^0(Z_s, \mathscr{O}_{Z_s}(m)) = P(m)$ for every $m \in \mathbb{Z}$ sufficiently large. This definition was given by Grothendieck, who also proved the following facts: $\mathrm{Hilb}^P_{\mathbb{P}^n}$ is indeed a sheaf, representable and in particular is represented by a projective scheme, also denoted $\mathrm{Hilb}^P_{\mathbb{P}^n}$ and called *Hilbert scheme*. A different, and easier, proof was given with some additional assumptions by Mumford, and was based on the Castelnuovo-Mumford regularity for the Grassmannian variety, using also a tool called flattening stratification; properness is proven using the valuative criterion and the criterion for flatness of $Z \to S$ when S is a smooth curve.

In the following to extend the concept of representability to stacks.

4.43 DEFINITION. A stack H over $\mathfrak{Sch}$ is *representable* if it is equivalent to a stack $\widetilde{\mathrm{h}}_X$ for some scheme X. A morphism of stack $\varphi\colon F \to G$ over $\mathfrak{Sch}$ is *(strongly) representable* if for every scheme S and for every morphism $\mathrm{h}_S \to G$, the stack $F \times_G \mathrm{h}_S$ is representable.

4.44 REMARK.

- Changing étale topology to smooth, or fppf, or fpqc topology does not change the result. Instead, Zariski topology is not enough.

- Let F be a representable stack, with an equivalence $\alpha\colon F \to \widetilde{\mathrm{h}}_T$ with T a scheme. Being α an equivalence, for every scheme S we have an equivalence $\alpha_S\colon F(S) \to \widetilde{\mathrm{h}}_T(S)$, but the target is a set, and we already saw that a groupoid is equivalent to a set if and only if it is rigid.

- As a corollary, F is representable if and only if:

  1. for every scheme S, $F(S)$ is a rigid groupoid, and

  2. the functor $\pi_0(F)\colon \mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Sets}$ is representable.

As before, we can define properties of representable morphisms of stacks from every property of morphisms of schemes which is stable under base change.

4.45 DEFINITION. If F and G are stacks, let $(F \times G)(S) := F(S) \times G(S)$ with obvious maps $F \times G \to F$ and $F \times G \to G$ (or equivalently, $F \times G := F \times_{\mathrm{h}_{\mathrm{Spec}\,K}}$

G). Then we can define the *diagonal morphism* $\Delta_F \colon F \to F \times F$ induced by

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}_F"] \arrow[d, "\mathrm{id}_F"'] & F \arrow[d] \\ F \arrow[r] & \operatorname{Spec} K. \end{tikzcd}$$

4.46 lemma. *If the diagonal morphism Δ_F is representable, then every morphism $\alpha\colon \mathrm{h}_X \to F$ with X a scheme is representable.*

Proof. Let $\beta\colon \mathrm{h}_S \to F$ a morphism with S a scheme; we have to prove that $\mathrm{h}_X \times_F \mathrm{h}_S$ is representable; we claim that there is a natural equivalence $\mathrm{h}_X \times_F \mathrm{h}_S \to (\mathrm{h}_X \times \mathrm{h}_S) \times_{F \times F} F$, where the morphism $F \to F \times F$ is the diagonal and the morphism $\mathrm{h}_X \times \mathrm{h}_S \to F \times F$ is (α, β). Then the target is representable by representability of Δ_F, so also the source is. $\square$

4.47 exercise. Prove the claim; more precisely:

1. prove the same for schemes: given morphisms of schemes $X \to Z$ and $Y \to Z$, there exists a natural isomorphism $X \times_Z Y \cong (X \times Y) \times_{Z \times Z} Z$;
2. prove it for groupoids;
3. the general case follows from the naturality of the previous step.

An easy but useful observation is that $\mathrm{h}_S \times \mathrm{h}_X$ is isomorphic to $\mathrm{h}_{S \times X}$ by definition.

4.48 remark. We are working with strong assumptions on schemes (of finite type over $K = \overline{K}$); to loosen these assumptions, we would still need some finiteness conditions on the diagonal morphisms.

4.49 definition. A *Deligne-Mumford algebraic stacks of finite type over an algebraically closed field* is a stack F such that:

1. Δ_F is representable;
2. there exists finitely many schemes (of finite type over the same field) S_i and morphisms $\alpha_i\colon \mathrm{h}_{S_i} \to F$ such that α_i is étale and $\coprod \alpha_i$ is surjective.

We could have just take one scheme S instead of finitely many S_i; but the choosen definition generalize better when we will loosen the assumptions.

4.50 definition. The 2*-category of Deligne-Mumford algebraic stacks*, denoted $\mathfrak{DMS}$ is defined as a full 2-subcategory of CFG over schemes or pseudofunctors $\mathfrak{Sch}^{\mathrm{opp}} \to \mathfrak{Groupoids}$.

4.51 lemma. *There is a fully faithful natural functor $\mathfrak{Sch} \to \mathfrak{DMS}$ which sends S to $\widetilde{\mathrm{h}_S}$.*

Thanks to the Lemma, we can embed schemes as a 1-subcategory of the 2-subcategory of DM-stacks. In particular, schemes are a 1-subcategory of rigid DM-stacks.

4.52 REMARK. Let F be a rigid DM-stack, that is, a DM-stack equivalent to a sheaf on $\mathfrak{Sch}$; as we anticipated, in general it is not true that F is representable. In particular, there is a notion of *algebraic spaces* due to Artin; the category of algebraic spaces is equivalent to the 1-category of rigid DM-stacks.

4.53 DEFINITION. An algebraic stack is called *weakly representable* if it is represented by an algebraic space. In the same way we can define *weakly representable* morphism of algebraic stacks: a morphism $\varphi\colon F \to G$ is weakly representable if for every $\mathrm{h}_S \to G$ with S scheme, $F \times_G \mathrm{h}_S$ is weakly representable.

Our aim will be extending to algebraic stacks as much as possible what we have for schemes and give examples.

4.54 REMARK. Any property of schemes which is local in the étale topology extends naturally to algebraic stacks. Here a local property in the étale topology is a property P such that a scheme S has P if and only if there exists an étale cover $\{\alpha_i\colon S_i \to S\}$ of S such that every S_i has P. Examples of étale local properties are being smooth, normal or reduced. Other étale locally properties are restrictions on the singularity type: Cohen-Macauley, Gorenstein, regular in some codimension.

So we can extend to algebraic stacks étale local properties of schemes, and to representable morphisms of algebraic stacks every property of morphisms of schemes that is stable under base change. Moreover, we want to know what properties of morphisms of schemes can be extended to every morphism of stacks.

4.55 DEFINITION. Let P a property of morphisms of schemes that is stable under base change; we say that P is *étale local in source and target* if for every morphism of schemes $f\colon X \to Y$ the following are equivalent:

1. f has P;
2. there exists an étale cover $\{X_i \to X\}$ such that for every i, $f \circ \alpha_i\colon X_i \to Y$ has P;
3. there exists an étale cover $\{Y_i \to Y\}$ such that for every i, $X \times_Y Y_i \to Y_i$ has P.

Note that since P is stable under base change, if the second statement is true, it is true for every étale cover; so it is for the third statement

4.56 EXERCISE. Let $f\colon X \to Y$ be a morphism, $\{Y_i \to Y\}$ an étale cover of Y, and $\{X_{i,j} \to X \times_Y Y_i\}$ an étale cover of $X \times_Y Y_i$ for every i. If P is a property étale local in source and target, then f has P if and only if $X_{i,j} \to Y_i$ has P for every i and j.

4.57 example. Properties that are étale local in source and target are smoothness of relative dimension n (in particular, étaleness, that is smooth with relative dimension 0), flatness, being unramified. Instead, properness, separatedness, surjectiveness, being an open embedding are not étale local in source and target.

4.58 exercise. Find counterexamples for properties that are not étale local in source and target.

4.59 exercise. Let P be a property of morphisms of schemes étale local in source and target; extend P to morphisms of algebraic stacks, defining f to have P if and only if there exists something such that for every something, something happens.

5 Examples of stacks

Lecture 7 (2 hours)
January 28th, 2009

5.1 *The stack* $\mathrm{B}\,G$

Let G be a finite group scheme (note that in characteristic zero, all group schemes are reduced and smooth). We define the stack $\mathrm{B}\,G$ as a CFG as follows:

$$\mathrm{B}\,G := \begin{cases} p \in \mathrm{Obj}\,(\mathrm{B}\,G) & \Leftrightarrow \quad p\colon \widetilde{S} \to S \text{ principal } G\text{-bundle}, \\ (\widetilde{f}, f) \in \mathrm{Mor}\,(p,q) & \Leftrightarrow \quad \begin{cases} \begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{f}} & \widetilde{T} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow[f]{} & T \end{array} \text{ commutes}, \\ \widetilde{f} \text{ is } G\text{-equivariant}. \end{cases} \end{cases}$$

Note that the objects of $\mathrm{B}\,G$ are principal G-bundle, or G-torsor, in the étale topology. We define the functor $\pi\colon\ \mathrm{B}\,G \to \mathfrak{Sch}$ sending p to the base scheme S and $(\widetilde{f}, f)$ to f. We can also translate this definition to pseudofunctors defining, for every $f\colon S \to T$, $f^*\widetilde{T} := \widetilde{T} \times_T S$ with the map $f^*\widetilde{T} \to S$.

5.1 exercise. Verify that the diagram

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\widetilde{f}} & \widetilde{T} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xrightarrow[f]{} & T \end{array}$$

in the example above is cartesian.

To check that $\mathrm{B}\,G$ is a stack, we have to prove the followings.

Morphisms are a sheaf. Given G-torsors $P \to S$ and $P' \to S$, we have the functor $M\colon (\mathfrak{Sch}/S)^{\mathrm{opp}} \to \mathfrak{Sets}$ that sends $f\colon S' \to S$ to $\mathrm{Mor}_{G\text{-torsor}}(f^\star P, f^\star P')$, and we have to prove that it is a sheaf.

So given $f\colon T \to S$ and an étale cover $\{g_i\colon T_i \to T\}$, defining

$$Q := f^\star P, \qquad Q_i := g_i^\star Q,$$
$$Q' := f^\star P', \qquad Q_i' := g_i^\star Q',$$

and having isomorphisms $\varphi_i\colon g_i^\star Q \to g_i^\star Q'$ such that $\varphi_i|_{T_{i,j}} = \varphi_j|_{T_{i,j}}$ (modulo canonical isomorphims), then $\{Q_i \to Q\}$ is an étale cover and

$$\psi_i\colon Q_i \xrightarrow{\varphi_i} Q_i' \xrightarrow{\text{nat}} Q'$$

are such that $\psi_i|_{Q_{i,j}} = \psi_j|_{Q_{i,j}}$. Now, since a scheme, and in particular h_Q, is a sheaf in the étale topology, there exists a unique $\psi\colon Q' \to Q$ inducing the ψ_i and the same for a morphism $\alpha\colon Q \to Q'$. What we want is to prove $\alpha = \psi^{-1}$: this is easy to check it on the Q_i, but again for the sheaf condition is true also globally. So from the φ_i we construct a morphism $\psi\colon Q' \to Q$ over the identity of T and we have to check that it is G-equivariant, but since G-equivariant means that the diagram

$$\begin{tikzcd} G \times Q' \arrow[r, "a"] \arrow[d, "\mathrm{id} \times \psi"'] & Q' \arrow[d, "\psi"] \\ G \times Q \arrow[r, "a"'] & Q \end{tikzcd}$$

commutes, and this is a local property, we can use the same argument. Notice that a similar argument applies whenever the objects are some kind of morphisms and the morphisms are cartesian diagrams.

Every descent datum is effective. A descent datum is a scheme S, an étale cover $\{S_i \to S\}$, G-torsors $\pi_i\colon P_i \to S_i$, isomorphisms $\varphi_{i,j}\colon P_i|_{S_{i,j}} \to P_j|_{S_{i,j}}$ such that on $S_{i,j,k}$ we have $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$. In practice, we blur the distinction between CFGs and pseudofunctors and often we ignore the natural isomorphisms. What we want is a G-torsor $P \to S$ with isomorphisms $\alpha_i\colon P|_{S_i} \to P_i$ inducing the $\varphi_{i,j}$. We assume firstly that $\{S_i \to S\}$ is actually a Zariski open cover; in this case, since $S_{i,j} \subseteq S_i$, we define $P_{i,j} := \pi_i^{-1} S_{i,j}$ and we have isomorphisms $\varphi_{i,j}\colon P_{i,j} \to P_{j,i}$ satisfying the cocycle condition; so we are done since with this data we know how to construct P and the α_i.

Assume for now that we can construct P and the α_i as scheme even if $\{S_i \to S\}$ is an étale cover; we need to give them a G-torsor structure, that is:

- define a morphism $a\colon G \times P \to P$;

- check that the diagram

$$\begin{tikzcd} G \times P \arrow[r, "a"] \arrow[dr] & P \arrow[d, "\pi"] \\ & S \end{tikzcd}$$

  commutes;

- check that a is an action;

- check that a makes $P \to S$ into a principal G-bundle.

To define a, we consider the composition $G \times P_i \to P_i \to P$ and we glue these maps using the fact that h_P is a sheaf. For the following two steps, we have to check that some morphisms are equal, and this is done, using the sheaf condition, by checking locally the equalities. Now, the P_i are principal G-bundles, so there exist étale covers $\{T_{i,k} \to S_i\}$ such that $P_i|_{T_{i,k}} \cong G \times T_{i,k}$ over $T_{i,k}$ in a G-equivarant way. But $T_{i,k} \to S_i \to S$ is étale (since it is the composition of two étale maps) and the $T_{i,k}$ covers S, so we have the étale local trivialization and $P \to S$ is a principal G-bundle.

Summing up, we have found that $\mathrm{B}\,G$ is a stack in the Zariski topology, and also in the étale topology up to prove the glueing condition. We will do it later.

5.2 exercise. Prove the uniqueness of the couple (P, α_i) up to something in the descent condition for $\mathrm{B}\,G$. Hint: based on the facts that schemes are sheaves in the étale topology.

Up to now, we found that $\mathrm{B}\,G$ is a stack; to prove algebraicity we have to prove the two conditions.

The diagonal is representable. Let S be a scheme and consider $\psi\colon \widetilde{\mathrm{h}}_S \to \mathrm{B}\,G \times \mathrm{B}\,G$. We define $F := \widetilde{\mathrm{h}}_S \times_{\mathrm{B}\,G \times \mathrm{B}\,G} \mathrm{B}\,G$ and we have to check that F is representable. The map ψ is just a couple of maps $\psi_1, \psi_2\colon \widetilde{\mathrm{h}}_S \to \mathrm{B}\,G$.

Consider a scheme T; firstly we have to find out what are the objects and the morphisms of $F(T)$. Its objects are triples $(f, p, (\alpha, \beta))$ with $f \in \widetilde{\mathrm{h}}_S(T)$ (that is, $f\colon T \to S$), $p\colon \widetilde{T} \to T$ a G-torsor, and $\alpha\colon f^\star P \to \widetilde{T}$, $\beta\colon f^\star P' \to \widetilde{T}$ isomorphisms as G-torsors. There are no morphisms from $(f, p, (\alpha, \beta))$ to $(f', p', (\alpha', \beta'))$ if $f \neq f'$ since there are no nontrivial morphisms in $\widetilde{\mathrm{h}}_S(T)$; if $f = f'$, a morphism is $\gamma\colon \widetilde{T} \to \widetilde{T}'$ such that $\alpha' = \gamma \circ \alpha$ and $\beta' = \gamma \circ \beta$. We should find out what F does to morphisms, but it is easy.

We can check first an easy necessary condition for F to be representable: $F(T)$ has to be rigid. In other words, we have to check that γ, when exists, is uniquely determined by source and target. Indeed, since α and α' are isomorphisms, the only possibility for γ is to be $\alpha' \circ \alpha^{-1}$. Notice that with the same argument, $\gamma = \beta' \circ \beta^{-1}$; therefore, there are no morphisms if $f \neq f'$ or $\alpha' \circ \alpha^{-1} \neq \beta' \circ \beta^{-1}$; there is just one morphism if $f = f'$ and $\alpha' \circ \alpha^{-1} = \beta' \circ \beta^{-1}$.

Since we want to prove that F is representable, we can choose to find a simpler but equivalent stack H and prove representability for H. So let

$$H(T) := \begin{cases} (f,\vartheta) \in \operatorname{Obj}(H(T)) & \Leftrightarrow \quad f\colon T \to S, \vartheta\colon f^\star P \xrightarrow{\sim} f^\star P' \text{ as } G\text{-torsor}, \\ \mathrm{id} \in \operatorname{Mor}((f,\vartheta),(f',\vartheta')) & \Leftrightarrow \quad \mathrm{f} = \mathrm{f}', \vartheta = \vartheta'. \end{cases}$$

We can send $F(T)$ to $H(T)$ by $(f,p,(\alpha,\beta)) \mapsto (f,\beta^{-1}\circ\alpha)$ and this is an equivalence of categories. For example, we can define an inverse $(f,\vartheta) \mapsto (f, f^\star P \to T, \mathrm{id}, \vartheta^{-1})$ or $(f,\vartheta) \mapsto (f, f^\star P' \to T, \vartheta, \mathrm{id})$. We should also check that this equivalence is compatible with pullbacks and with the natural 2-equivalence for each pair of composable maps in $\mathfrak{Sch}$.

Now, consider a scheme S, two G-torsors $P \to S$ and $P' \to S$; we have to prove that the functor $N\colon (\mathfrak{Sch}\,/S)^{\mathrm{opp}} \to \mathfrak{Sets}$ with

$$N(f) := \operatorname{Mor}_{G-torsor}(f^\star P, f^\star P')$$

is representable, that is, isomorphic to some h_U. In other words, we want a morphism $u\colon U \to S$ and an isomorphism $\vartheta\colon u^\star P \to u^\star P'$ such that every $\beta\colon f^\star P \to f^\star P'$ is induced by a unique morphism $T \to U$.

5.3 EXERCISE. Prove that N is representable. Hint: whatever U is, it has to admit a G-action. For, if G is abelian, given $\lambda \in N(f)$, that is $\lambda\colon f^\star P \to f^\star P'$, we can compose λ with multiplication by an element $g_0 \in G$ and obtain another element of $N(f)$. Second hint: think of $P \times_S P'$.

5.4 EXERCISE. Let $\pi\colon P \to S$ be a principal G-bundle; then $\pi^\star P$ is canonically trivial.

THERE IS AN ÉTALE COVER BY REPRESENTABLE STACKS. We consider a morphism $\tilde{\mathrm{h}}_S \to \mathrm{B}\,G$, that is, a principal G-bundle $P \to S$, and a morphism $\operatorname{Spec} K \to \mathrm{B}\,G$, that is, the trivial principal G-bundle $G \times \operatorname{Spec} K \to \operatorname{Spec} K$; call H the fiber product. An object in $H(T)$, since an object in $(\operatorname{Spec} K)(T)$ is trivial, is just a couple (f,α) with $f\colon T \to S$ and $\alpha\colon f^\star P \xrightarrow{\sim} G \times T$. There are no morphism from (f,α) to (g,β), if $f \neq g$ or $\alpha \neq \beta$; there is only the identity if $f = g$ and $\alpha = \beta$. This is so since to have morphisms, the diagram

$$\begin{tikzcd} f^\star P \arrow[rr, "\mathrm{id}^\star"] \arrow[dr, "\alpha"'] & & f^\star P \arrow[dl, "\beta"] \\ & G \times T & \end{tikzcd}$$

have to commute. We establish a map $\mathfrak{Sch}\,/P \to H$ sending $g\colon T \to P$ to $(f := g \circ \pi, g^\star\beta)$, where $\beta\colon \pi^\star P \to G \times P$ is the canonical isomorphism of the previous exercise.

5.5 EXERCISE. Check that the map is bijective, that is, H is represented by P.

So far, we have proven the existence of a 2-cartesian diagram of stacks

$$\begin{tikzcd} \widetilde{\mathrm{h}}_P \arrow[r] \arrow[d] & \widetilde{\mathrm{h}}_{\operatorname{Spec} K} \arrow[d] \\ \widetilde{\mathrm{h}}_S \arrow[r] & \mathrm{B}\,G \end{tikzcd}$$

for any $\widetilde{\mathrm{h}}_S \to \mathrm{B}\,G$ defined by a principal G-bundle P over S.

5.6 notation. From now on, we will write S for a scheme, for its functor of points h_S and for the associate stack $\widetilde{\mathrm{h}}_S$. In other words, we identificate schemes with a full subcategory of algebraic stacks. For example, the previous diagram becomes

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\ S \arrow[r] & \mathrm{B}\,G. \end{tikzcd}$$

Assume that G is finite and étale over $\operatorname{Spec} K$, and that $\pi\colon P \to S$ is étale. Since π is a G-torsor, it is surjective, so in particular $\operatorname{Spec} K \to \mathrm{B}\,G$ is étale and surjective because π is a base change of $\operatorname{Spec} K \to \mathrm{B}\,G$. In this particular case, this alone is already an étale cover by representable stacks and completes the proof of the second condition. Note that for the complex number, every finite group is étale over $\operatorname{Spec} K$ since every point has the trivial $\operatorname{Spec} K$ structure.

6 Moduli of curves

Lecture 8 (2 hours)
February 4th, 2009

6.1 definition. Let $g, n \in \mathbb{Z}$ such that $g, n \geq 0$ and $2g - 2 + n > 0$ (this last condition excludes $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$). We define a category $M_{g,n}$ over schemes in this way:

$$M_{g,n} := \begin{cases} \begin{array}{c} C \\ {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \sigma_1,\ldots,\sigma_n} \\ S \end{array} \in \operatorname{Obj}\left(M_{g,n}\right) & \Leftrightarrow \quad \begin{array}{l} \pi \text{ flat projective morphism,} \\ \sigma_i \text{ sections of } \pi, \\ \text{for every } s \in S \text{ (that is, } \operatorname{Spec} K \to S\text{),} \\ C_s \text{ is a smooth connected genus } g \text{ curve,} \\ \sigma_1(s), \ldots, \sigma_n(s) \text{ are distinct points.} \end{array} \\[2ex] (\tilde{f}, f) \in \operatorname{Mor}\left(\begin{array}{c} C \\ {\scriptstyle \pi}\downarrow\uparrow{\scriptstyle \sigma_1,\ldots,\sigma_n} \\ S \end{array}, \begin{array}{c} C' \\ {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle \sigma'_1,\ldots,\sigma'_n} \\ S' \end{array} \right) & \Leftrightarrow \quad \begin{tikzcd} C \arrow[r, "\tilde{f}"] \arrow[d, shift left, "\pi"] \arrow[dr, phantom, "\square"] & C' \arrow[d, shift right, "\pi'"'] \\ S \arrow[r, "f"'] \arrow[u, shift left, "\sigma_1,\ldots,\sigma_n"] & S' \arrow[u, shift right, "\sigma'_1,\ldots,\sigma'_n"'] \end{tikzcd} \end{cases}$$

This category is called the *stack of n-pointed smooth genus g projective curves*. If $n = 0$, we write M_g for $M_{g,n}$.

6.2 exercise. Define composition of morphisms in $M_{g,n}$.

6.3 exercise. The category $M_{g.n}$ is a CFG over schemes (note that this is true also for the four values of (g, n) we excluded).

6.4 proposition. *The CFG $M_{g,n}$ is a stack.*

Proof. We have to show that morphisms are a sheaf and that every descend datum is effective.

1. Fix a scheme S and two objects

$$\xi = \begin{array}{c} C \\ \pi \downarrow \uparrow \sigma_1, \dots, \sigma_n \\ S \end{array}, \qquad \xi' = \begin{array}{c} C' \\ \pi \downarrow \uparrow \sigma'_1, \dots, \sigma'_n \\ S \end{array}.$$

We define the functor $H \colon \mathfrak{Sch}/S \to \mathfrak{Sets}$ which sends $g \colon T \to S$ to $\mathrm{Mor}(g^\star\xi, g^\star\xi')$. We define $C_T := C \times_S T$ and $C'_T := C \times_{S'} T$, $\pi_T \colon C_T \to T$ and $\pi'_T \colon C'_T \to T$ induced by the cartesian product, $\sigma_{i,T} \colon T \to C_T$ and $\sigma'_{i,T} \colon T \to C'_T$ by the universal property applied to id_T and $\sigma_i \circ g$ and $\sigma'_i \circ g$ respectively.

We note that a morphism $g^\star\xi \to g^\star\xi'$ is a morphism $\tilde{g} \colon C_T \to C'_T$ such that $\sigma_{i,T} = \sigma_{i,T}{}' \circ \tilde{g}$, $\pi_T = \pi'_T \circ \tilde{g}$ and $\tilde{g}$ makes the diagram over the identity cartesian, that is equivalent to ask $\tilde{g}$ to be an isomorphism. Consider now an étale cover $\{T_i \to T\}$ and isomorphisms $\tilde{g}_i \colon C_{T_i} \to C'_{T_i}$ such that $\tilde{g}_i|_{C_{T_{i,j}}} = \tilde{g}_j|_{C_{T_{i,j}}}$ where equality means natural isomorphism. Now we have a commutative diagram

$$\begin{tikzcd}[column sep=small]
C_{T_{i,j}} \arrow[r, "\sim"] \arrow[d, "\wr"] & C_{T_j}|_{T_{i,j}} \arrow[dr, "\tilde{g}_j|_{T_{i,j}}"] & \\
C_{T_i}|_{T_{i,j}} \arrow[dr, "\tilde{g}_i|_{T_{i,j}}"'] & & C'_{T_j}|_{T_{i,j}} \\
& C'_{T_i}|_{T_{i,j}} & C'_{T_{i,j}} \arrow[l, "\sim"'] \arrow[u, "\wr"']
\end{tikzcd}$$

and since $\mathrm{h}_{C'_T}$ is a sheaf, the $\tilde{g}_i$ glue to a unique morphism $\tilde{g} \colon C_T \to C'_T$, because $\{C_{T_i} \to C_T\}$ is an étale cover. We need to prove the followings:

- $\tilde{g}$ is an isomorphism;
- it commutes with $\sigma_{i,T}$ and $\sigma'_{i,T}$;
- it commutes with π_T and π'_T.

Each of these properties is true étale locally on the target, so the last two are true globally since morphisms are a sheaf in the étale topology, and the first is true because we can define $\tilde{g}^{-1}$ locally and then glue.

2. Let S be a scheme and $\{S_i \to S\}$ and étale cover; moreover let ξ_i objects $C_i \to S_i$ with isomorphisms $\varphi_{i,j} \colon C_i|_{S_{i,j}} \to C_j|_{S_{i,j}}$. We want to glue the ξ_i to a global ξ over S; we can construct the map $C \to S$ by descent theory;

to construct the sections, we note that composing with $C_i \to C$ we have maps $S_i \to C_i \to C$ which agree locally, so we use the fact that h_C is a sheaf to construct global σ_i. We have to prove that:

- π is smooth and projective;
- σ_i are sections;
- every fiber is a genus g smooth connected curve.

As for the first, smoothness and properness are local in the target even in the Zariski topology; we will point out projectivity later. The last condition is trivial, since a fiber of ξ is a fiber in at least one of the ξ_i (this is so since being an étale cover, for schemes over a fixed algebraically closed ground field, implies the property that every morphism $\operatorname{Spec} K \to S$ factors through at least one $S_i \to S$). ☐

6.5 theorem.

1. *The stack $M_{g,n}$ is a smooth irreducible algebraic DM-stack of dimension $3g - 3 + n$.*

2. *It has a natural compactification $\overline{M}_{g,n}$ that is smooth and irreducible given by some similar definition.*

This theorem is proved in the paper of Deligne and Mumford and was the first reason to define algebraic stacks.

6.6 lemma.

1. *$M_{0,3}$ is representable by* $\operatorname{Spec} K$*;*

2. *$M_{0,n}$ is representable by a smooth quasi-projective connected rational variety of dimension $n - 3$ for $n \geq 3$.*

Proof. Let C be a smooth projective genus 0 curve; then C is isomorphic to $\mathbb{P}^1$ and given distinct points $x_1, x_2, x_3 \in C$, there exists a unique isomorphism $f\colon C \to \mathbb{P}^1$ such that $f(x_1) = 0$, $f(x_2) = 1$, $f(x_3) = \infty$. So the unique point $\operatorname{Spec} K \to M_{0,3}$ is the map $\mathbb{P}_1 \to \operatorname{Spec} K$ with the three sections 0, 1, ∞. To prove that $M_{0,3}$ is $\operatorname{Spec} K$, we have to prove that for every object $C \to S$ in $M_{0,3}$ over S, we have a unique morphism $(\tilde{f}, \mathrm{id}_S)$ from an object ξ over S to the trivial $\mathbb{P}^1 \times S \to S$. Fiberwise this is precisely the first remark. In general, we consider $s_1(S) \subseteq C$: it is a Cartier divisor, so we can consider $\mathscr{L} := \mathscr{O}_C(s_1(S))$.

We are interested in the push-forward $\pi_\star \mathscr{L}$; we know that for every cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

there exists a natural morphism $\vartheta\colon g^\star p_\star \mathscr{F} \to q_\star f^\star \mathscr{F}$ for every coherent sheaf $\mathscr{F}$ over X. If we assume p to be flat and projective, $\mathscr{F}$ flat over Y, and $Y' = \operatorname{Spec} K$, then $R^i q_\star f^\star \mathscr{F} = \mathrm{H}^i(X', \mathscr{F}|_{X'})$.

We state a special case of Theorem 3.12.11 in [Har77]: if $y \in Y$ is the image of $Y' = \operatorname{Spec} K$, and $\mathrm{H}^i(X_y, \mathscr{F}_{X_y}) = 0$ for $i > 0$, then $R^i p_\star \mathscr{F} = 0$ for $i > 0$ and $p_\star \mathscr{F}$ is locally free on Y and the natural map $\mathrm{H}^0(X_y, \mathscr{F}_{X_y}) \to g^\star(p_\star \mathscr{F})$ is an isomorphism.

Using this special case in our situation, we get that on a fiber $C_s \cong \mathbb{P}^1$, $\mathscr{L}|_{C_s} = \mathscr{O}_{\mathbb{P}^1}(1)$ and so $\mathrm{H}^1(C_s, \mathscr{L}|_{C_s}) = 0$ and $\mathrm{H}^0(C_s, \mathscr{L}|_{C_s})$ is 2-dimensional, so its projectivization can be canonically identified with C_s.

In particular we get a map $\mathbb{P}(\pi_\star \mathscr{L}) \to S$ and $C \to \mathbb{P}(\pi_\star \mathscr{L})$ commuting with $C \to S$. Now, $\pi_\star \mathscr{L}(S) = \mathscr{L}(C)$ has a canonical section; if $\mathscr{L}_1 = \mathscr{L}$, and $\mathscr{L}_2, \mathscr{L}_3$ are the same with the other sections, it turns out that $\pi_\star(\mathscr{L}_1 \otimes \mathscr{L}_2^\vee) = \mathscr{O}_S$ and we found a trivialization of $\pi_\star \mathscr{L}$.

In general, for $n \geq 3$, we have a unique map $(\mathbb{P}^1, x_1, \ldots, x_n)$ to $(\mathbb{P}^1, 0, 1, \infty, y_4, \ldots, y_n)$, so as a set

$$M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \text{diagonals}.$$

It can be proved it is affine. □

Note that in case $M_{0,2}$, we have a lot of automorphisms: it can be proved that it is equivalent to the stack $\mathrm{B}\mathbb{G}_m$ (that is the stack of invertible sheaves with their automorphisms) via $\mathscr{L} \in \operatorname{Pic}(S) \to (\mathbb{P}(\mathscr{O}_S \oplus \mathscr{L}) \to S)$ with the natural sections 0 and ∞. Also $M_{1,0}$ has a lot of automorphisms: since a genus 1 curve E is an abelian group, $E \subseteq \operatorname{Aut}(E)$.

6.7 DEFINITION. The *stack of elliptic curves* is $M_{1,1}$. An *elliptic curve* is a pair $(E, 0)$ where E is a smooth projective genus 1 curve and $0 \in E$.

We can describe $M_{1,1}$ in the holomorphic way (see Chapter 4 in [Har77]) or in the algebraic way.

6.8 PROPOSITION. *Let (E, p_0) be an elliptic curve over a ground field with characteristic different from 2 and 3; then there exists $\varphi\colon E \to \mathbb{P}^2$ such that:*

1. *φ is a closed embedding;*
2. *$\varphi(p_0) = (0, 1, 0)$;*
3. *$\varphi^\star \mathscr{O}_{\mathbb{P}^1}(1) = \mathscr{O}_E(3p_0)$;*
4. *$\varphi(E)$ is the locus described by $y^2 = x^3 + ax + b$ for some $a, b \in K$ (Weierstrass normal form).*

Proof. A morphism $\varphi\colon E \to \mathbb{P}^2$ is an invertible sheaf $\mathscr{L}$ over E (such that $\mathscr{L} = \varphi^\star \mathscr{O}_{\mathbb{P}^2}(1)$) and three sections $x, y, z \in \mathscr{L}(E)$. By the requests, $\mathscr{L} = \mathscr{O}_E(3p_0) = \mathscr{O}_E(p_0)^{\otimes 3}$. There exists a natural section $s \in \mathscr{O}_E(p_0)$ vanishing on p_0; we take $z = s^{\otimes 3}$, since the zero locus of z is p_0 with multiplicity 3. Since Ω^1_C is trivial for genus 1 curve, we get that if $\deg \mathscr{L} > 0$ (so that $\deg \mathscr{L}^\vee < 0$), then $\deg \mathscr{L}^\vee \otimes \Omega^1_C < 0$ and so $h^0(\mathscr{L}^\vee \otimes \Omega^1_C = 0$; using Riemann-Roch, we get $h^0(\mathscr{L}) = d$ every time $d > 0$. Now, $H^0(\mathscr{O}_C(2p_0))$ contains $s^{\otimes 2}$ so we can choose another section t such that $(s^{\otimes 2}, t)$ is a basis; in the same way, we choose a section $u \in H^0(\mathscr{O}_C(3p_0))$. Then (st, u, s^3) gives a basis such that almost all condition are satisfied, but the normal form of the equation. The trick is to choose u and t in such a way they satisfy some simmetry with respect to the involution. □

Lecture 9 (2 hours)
February 9th, 2009

We have seen that if E is a smooth genus 1 curve with a marked point p_0, we have a somehow natural basis for $H^0(E, 3p_0)$, composed by:

- the third power of the section $s \in H^0(E, p_0)$;
- the product of a "special" section t of $H^0(E, 2p_0)$ with s;
- another "special" section u of $H^0(E, 3p_0)$.

We can choose z to be s^3, x to be st and y to be u; in this way, choosing cleverly t and u, the embedding of E into $\mathbb{P}^2$ given by x, y, z is the locus $y^2 = x^3 + ax + b$ for some $a, b \in K$ (in affine coordinates). In particular, p_0 is sent to $(0, 1, 0)$.

We want to know how unique is this equation. In other words, what are the isomorphisms $\alpha\colon C_{a,b} \to C_{a_1,b_1}$ such that $(0, 1, 0)$ is fixed. It turns out that these isomorphisms must be linear (that is, they come from automorphisms of $\mathbb{P}^2$); moreover, by the choice of x, y, z and by the fixed point, α has the form $\alpha(x, y, z) = (\lambda^2 x, \lambda^3 y, z)$. We call α_λ such an α. In the equation, the first two terms are multiplied by λ^6 when α_λ acts, the third by λ^2 and the last is unchanged; so $a_1 = \lambda^4 a$ and $b_1 = \lambda^6 b$.

6.9 definition. If $a_0, \dots, a_n > 0$, the *weighted projective stack* with weights $a_0, \dots, a_n$ is the quotient $[\mathbb{A}^{n+1} \setminus 0/\mathbb{G}_m]$ where the action is given by

$$\lambda(x_0, \dots, x_n) := (\lambda^{a^0} x_0, \dots, \lambda^{a_n} x_n).$$

6.10 theorem. *There is an isomorphism of stacks $\mathbb{P}(4, 6) \setminus \{p\} \to M_{1,1}$.*

6.11 definition. More generally, if X is a scheme and G is a group scheme acting on X, the stack quotient $[X/G]$ is defined via

$$[X/G](S) = \left\{ \begin{array}{l} P \to X \\ \downarrow \\ S \end{array} \middle| \begin{array}{l} P \to S \text{ principal } G\text{-bundle} \\ P \to X\ G\text{-equivariant} \end{array} \right\}$$

where the pullback is the principal G-bundles' one and the isomorphisms are isomorphisms of principal G-bundle commuting with the maps to X.

Modulo issues of descending principal G-bundles, depending on what kind of groups we allow, we have the following.

6.12 proposition.

- *$[X/G]$ is a stack;*

- *we have a map $X \to [X/G]$;*
- *for any morphism $S \to [X/G]$ (corresponding to a principal G-bundle $P \to S$ with a G-equivariant map to X), the diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & {[X/G]} \end{tikzcd}$$

is 2-cartesian;

- $\Delta_{[X/G]}$ *is representable.*

Tipical groups what pose no problems are $\mathrm{GL}(n)$, $\mathbb{P}\,\mathrm{GL}(n)$, $\mathrm{SL}(n)$ their products, finite group (with care if we are in positive characteristic). Problematic groups are $\mathbb{A}^1$, elliptic curves. Altough, for weighted projective stacks we are safe since we can do all computations explicitly.

6.13 PROPOSITION. *The stack $\mathbb{P}(a_0, \ldots, a_n)$ is a DM-algebraic stack.*

6.14 EXERCISE. Prove the two conditions in characteristic 0. To prove the second one, we may use as étale cover the usual affine charts U_i. More precisely, consider $V_i \subseteq \mathbb{A}^{n+1} \setminus \{0\}$ defined by $x_i \neq 0$ and prove that $[V_i \setminus \mathbb{G}_m] \to \mathbb{P}(a_0, \ldots, a_n)$ is representable and an open embedding. Now, let $U_i \subseteq V_i$ defined by $x_i = 1$; V_i is $\mathbb{G}_m$-invariant, but U_i is not; but it is fixed by the a_i-th roots of unity, so it has a natural μ_{a_i} action. Prove that there is an isomorphism $[U_i/\mu_{a_i}] \cong [V_i/\mathbb{G}_m]$.

The corrispondence is given in this way: consider an object $\xi \in [V_i/\mathbb{G}_m](S)$, that is a principal $\mathbb{G}_m$-bundle $\pi\colon P \to S$ with a map $f\colon P \to V_i$; define $Q := f^{-1}(U_i)$; show that the $\mathbb{G}_m$ action on P induces a μ_{a_i}-action on Q making it into a principal μ_{a_i}-bundle; in particular we have to show that:

- Q is μ_{a_i}-invariant;
- to show that Q is principal, we can assume P is trivial.

So now we have

$$U_i \to [U_i/\mu_{a_i}] \cong [V_i/\mathbb{G}_m] \to \mathbb{P}(a_0, \ldots, a_n).$$

We proved that the last is a representable open embedding; we can prove that the first is a representable surjective étale morphism. This is so since a μ_{a_i}-bundle is surjective and étale, indeed given $S \to [U_i/\mu_{a_i}]$ we have a diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & U_i \arrow[d] \\ S \arrow[r] & {[U_i/\mu_{a_i}]} \end{tikzcd}$$

that is cartesian and in particular $Q \to U_i$ induces $S \to [U_i/\mu_{a_i}]$. Note that here we need characteristic 0 since $\mu_n \to \operatorname{Spec} K$ is étale if and only if n is prime to $\operatorname{ch} K$. The last thing to prove is that the images of $[V_i/\mathbb{G}_m] \to \mathbb{P}(a_0, \dots, a_n)$ cover the target.

6.15 remark. In the solution of the exercise we do not need to use properties of $\mathbb{G}_m$. So, if a group scheme G acts on a scheme X and U is a G-invariant open subscheme of X, then $[U/G] \to [X/G]$ is a representable morphism and an open embedding. More complicated is that if we have a morphism of scheme $X \to Y$ that is G-equivariant, then $[X/G] \to [Y/G]$ is representable and has all the properties of $X \to Y$ that are étale-local in the target.

6.16 remark. Let X be a scheme and G a group scheme acting on X, then we have a map $[X/G] \to [\operatorname{Spec} K/G] = \mathrm{B}\,G$. If $P \to X$ is a G-torsor, it is easy to prove that $[P/G] \cong X$; so we have a map $X \to \mathrm{B}\,G$ corresponding to the G-torsor $P \to X$. This map gives us information about X. Thanks to the canonical correspondance between $\mathrm{GL}(n, K)$-torsor on X with fiber bundle of rank n, to every fiber bundle $E \to X$ of rank n we have a map $X \to \mathrm{B}\,\mathrm{GL}(n, K)$. It can be proven that $\mathrm{H}^\star(\mathrm{B}\,\mathrm{GL}(n, K), \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$ where c_i are n free generators and $\deg c_i = 2i$. Moreover the pullback of c_i is

If G is a good algebraic group acting on a scheme X with étale stabilizers (that is, in characteristic 0, G_x is finite; in the holomorphic category discrete is enough), then $[X/G]$ is a DM-algebraic stack. The key point is Luna's étale slice theorem.

We return now to the problem of the isomorphism between elliptic curves and $\mathbb{P}(4, 6)$ minus a point.

6.17 exercise. Given a, b, then $C_{a,b}$ is smooth if and only if $f(t) = t^3 + at + b$ has 3 distinct point. Note that such a polynomial cannot have three equal non-zero roots, since the sum of them is 0; if the roots are $t_0, t_0, -2t_0$, then $b = 2t_0^3$ and $a = -3t_0^2$ and they parametrize a cuspidal cubic.

In other words, the locus $V = (4a^2 + 27b^3 \neq 0) \subseteq \mathbb{A}^2$ is open and $\mathbb{G}_m$-invariant; we establish a map $V \to M_{1,1}$, corresponding to a family $\pi\colon E \to V$ with a section $s\colon V \to E$; in particular, we choose $E \subseteq V \times \mathbb{P}^2$, with coordinates (a, b) on V and (x, y, z) on $\mathbb{P}^2$ and E is the zero locus of $zy^2 - x^3 - axz^2 - bz^3$; we define the section as $s(a, b) := ((a, b), (0, 1, 0))$. The action of $\mathbb{G}_m$ on E is lifted from the action on V: $\lambda((a, b), (x, y, z)) := ((\lambda^4 a, \lambda^6 b), (\lambda^2 x, \lambda^3 y, z))$.

6.18 exercise. Get an induced morphism $[V/\mathbb{G}_m] \to M_{1,1}$.

To prove it is an isomorphism, we can explicitly construct the inverse: from a family $\pi\colon E \to S$ with a section s, we construct the line bundle $\mathscr{F} := \pi_\star \mathscr{O}_E((s(S))^{\otimes 3})$ and we associate to it a principal $\mathbb{G}_m$ bundle on S given by the triples inside the frames for $\mathscr{F}$. In this way we get a principal $\mathbb{G}_m$-bundle $P \to S$; pullbacking E to P we get $E_P \to P$ and by the natural embedding $E_P \to \mathbb{P}^2$ an induced morphism $P \to V$. This is doable, but there are simpler way.

6.19 REMARK. If the characteristic of K is 2 or 3, it is not true anyore that $M_{1,1}$ is an open embedding in $\mathbb{P}(4,6)$ (in particular it is not true that $\mathbb{P}(4,6)$ is a DM-algebraic stack); but $M_{1,1}$ is still a DM-algebraic stack and indeed $M_{g,n}$ is a DM-algebraic stack, smooth of dimension $3g-3+n$ (even over $\operatorname{Spec}\mathbb{Z}$).

6.20 REMARK. Instead, in the complex analytic approeach, if E is a genus one Riemann surface, then its universal cover is isomorphic to $\mathbb{C}$; moreover, there exists a discrete lattice $\Lambda \subseteq \mathbb{C}$ isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ such that $E \cong \mathbb{C}/\Lambda$. Up to a $\mathbb{C}$-linear change of coordinates, we may assume that $\Lambda = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot\tau$ with $\Im\tau > 0$; τ is unique up to an action of the group $\mathrm{SL}(2,\mathbb{Z})$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) := \frac{a+b\tau}{c+d\tau}.$$

Note that $-I$ acts trivially on τ. In a similar way, we can identify $M_{1,1}$ with $[\mathbb{H}/\,\mathrm{SL}(2,\mathbb{Z})]$.

On the algebraic side, it seems natural to consider the whole $\mathbb{P}(4,6)$ as a compactification $\overline{M}_{1,1}$ of $M_{1,1}$. This is reasonable, if we allow nodal elliptic curves. Indeed, if $C_{a,b}$ is singular, it is a nodal curve where p_0 is the point at infinity that is smooth. We have now to redefine $M_{1,1}$ to allow such curves; it turns out that we have to change the condition on the fibers: in particular, to change smooth of genus 1 with at most nodal, with arithmetic genus 1 (because the arithmetic genus is constant on flat families) with section on a smooth point. Note that we allow nodal curves but no cusps.

6.21 EXERCISE. If $C, C' \subseteq \mathbb{P}^2$ are degree 3 irreducible curves with one node and no other singularities, then for all choice of $p \in C$ and $p' \in C'$ smooth points, there exists an isomorphism $\pi\colon C \to C'$ such that $\pi(p) = p'$. Moreover, such isomorphisms are precisely two. This means that we are allowing just one more fiber, up to isomorphisms.

Note that we insisted on asking irreducibility. For example, given a smooth genus 1 curve, with a rational component attaching to it on a point, with a marked point on the rational component, it has again arithmetic genus 1; what is wrong is that the automorphisms group of this not irreducible curve has positive dimension.

Viewed on the analytic side, $\mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$ is generated by two elements $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ which sends τ to $\tau+1$ and $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ which sends τ to $-\tau^{-1}$. So to have a domain of the action is to take the strip $-1/2 \leq \Re z \leq 1/2$ with $|z| \geq 1$ and $\Im z > 0$. In particular there are two points which give lattices with additional simmetries: $\tau = i$ gives an additional simmetry of degree 4 and $e^{2\pi i/3}$ gives a simmetry of degree 6. These are just the points with additional automorphisms in $\mathbb{P}(4,6)$. So the point we are adding compactifying $M_{1,1}$ is not a special point of $\mathbb{P}(4,6)$ but a point with just the ordinary $\mathbb{Z}_2$ automorphism group. From the analytic viewpoint, we are adding a point of infinity; this is reasonable since the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{P}^1$ is the restriction of the action of $\mathrm{GL}(2,K)$. In other words, modulo the transformation of the half plane to the

Poincaré disc, we are adding the origin of the disc.

Lecture 10 (2 hours)
February 10th, 2009

6.22 remark. If X is a DM-algebraic stack, and $x \in \mathrm{Obj}(X(K))$, then $\mathrm{Aut}_{X(K)}(x)$ is a finite étale group. In other words, if we consider the morphism $x\colon \operatorname{Spec} K \to X$ and the fiber product $A := \operatorname{Spec} K \times_X \operatorname{Spec} K$, then thanks to algebraicity of X, A is a scheme and:

1. $A(K)$ is naturally bijective to $\mathrm{Aut}_{X(K)}(x)$ (this gives to $\mathrm{Aut}_{X(K)}(x)$ the scheme structure);

2. $A \to \operatorname{Spec} K$ is étale.

6.23 exercise. Use the universal property of the fiber product to define a group structure on A.

Proof.

1. By definition, $A(K) = (\operatorname{Spec} K)(K) \times_{X(K)} (\operatorname{Spec} K)(K)$; since $(\operatorname{Spec} K)(K)$ has a unique object with only the identity, $A(K)$ is naturally a set with objects isomorphisms from the image of the first $(\operatorname{Spec} K)(K)$ to the image of the second, that is, automorphisms of x. Note that here we did not use the fact that X is an algebraic stack of Deligne Mumford.

2. By assumption, there exists an étale cover $\{\pi_i\colon U_i \to X\}$, that is, for every $S \to X$ with S a scheme, $\{U_i \times_X S \to S\}$ is an étale cover of S. Applying this to $\operatorname{Spec} K \to X$, we get that $\{\widetilde{U_i} := U_i \times_X \operatorname{Spec} K \to \operatorname{Spec} K\}$ is an étale cover. In particular, there exists a morphism $\operatorname{Spec} K \to \widetilde{U_i}$ such that the composition is the identity. So we have the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[dr] \arrow[drr, dashed, bend left] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \widetilde{U_i} \arrow[r] \arrow[d] & U_i \arrow[d] \\
& \operatorname{Spec} K \arrow[r] & X
\end{tikzcd}$$

and we got a well defined dashed map $i\colon \operatorname{Spec} K \to U_i$ commuting with the other maps. Now we consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d] \\
\widetilde{U_i} \arrow[r] \arrow[d] & U_i \arrow[d] \\
\operatorname{Spec} K \arrow[r] & X
\end{tikzcd}$$

where both square are cartesian. Since $U_i \to X$ is étale, so is $\widetilde{U_i} \to \operatorname{Spec} K$; then $\widetilde{U_i} \to U_i$ is a disjoint union of finitely many copies of $\operatorname{Spec} K$.

Now, if $y\colon \operatorname{Spec} K \to U_i$ is a point of U_i, then the fiber product of the maps y and x is $\operatorname{Spec} K$ if $x = y$, and is empty otherwise. Then, since A is a disjoint union of such fiber product, it is a disjoint union of copies of $\operatorname{Spec} K$. $\square$

Now we want to generalize $\overline{M}_{1,1}$ to other g and n. So we define $\mathfrak{M}_{g,n}$ to be the stack associated to the pseudofunctor

$$\mathfrak{M}_{g,n}(S) := \begin{cases} \begin{matrix} C \\ \pi\downarrow\uparrow\sigma_1,\dots,\sigma_n \\ S \end{matrix} \in \operatorname{Obj}\left(\mathfrak{M}_{g,n}(S)\right) & \Leftrightarrow \quad \begin{array}{l} \pi \text{ flat and proper,} \\ \sigma_i \text{ section of } \pi, \\ \text{for every } s \in S(K),\ C_s \text{ is a nodal} \\ \text{connected curve of arithmetic genus } g, \\ \text{with } \sigma_i(s) \text{ are distinct and smooth points;} \end{array} \\ \varphi\colon C \xrightarrow{\sim} C' \in \operatorname{Mor}\left(\begin{matrix} C \\ \pi\downarrow\uparrow\sigma_1,\dots,\sigma_n \\ S \end{matrix}, \begin{matrix} C' \\ \pi'\downarrow\uparrow\sigma'_1,\dots,\sigma'_n \\ S \end{matrix} \right) & \Leftrightarrow \quad \varphi \text{ commutes with all other structures.} \end{cases}$$

We call a curve $(C, x_1, \dots, x_n) \in \operatorname{Obj}(\mathfrak{M}_{g,n}(\operatorname{Spec} K))$ a *prestable genus g curve*.

There are several approaches to define the arithmetic genus of a curve; the first is to define it as $\mathrm{h}^1(C, \mathscr{O}_C)$; we could also define it as $\mathrm{h}^0(C, \omega_C)$ where ω_C is the dualizing sheaf; this works if the curve is not to singular, in particular if it is Gorenstein (that precisely means that C has a dualizing sheaf and it is invertible); note that all nodal curves are Gorenstein.

6.24 EXERCISE. Consider a closed embedding i of a curve C into a smooth surface S; then $\omega_C = K_S \otimes \mathscr{O}_S(C)|_C$, where $K_S := \bigwedge^2 \Omega_S$. Check this formula in the case C is smooth.

A third way to define the arithmetic genus is due to the following lemma.

6.25 LEMMA. *Let $\pi\colon C \to S$ be a flat proper morphism with connected nodal curves as fibers. Then the arithmetic genus of the fibers is locally constant.*

Proof. There exists a relative dualizing sheaf ω_π on C such that $\omega_\pi|_{C_s} \cong \omega_{C_s}$. By assumption, ω_π is a line bundle and so is flat over S (because flatness is a local property and locally it is just the structural sheaf). Now, flatness and properness of π implies that $\chi(C_s, \omega_\pi|_{C_s})$ is locally constant. But this characteristic is just $\mathrm{h}^0(C_s, \omega_{C_s}) - \mathrm{h}^1(C_s, \omega_{C_s})$. The last term is $\mathrm{h}^0(C_s, \mathscr{O}_{C_s}) = 1$ by properness and so the arithmetic genus (the first term) is locally constant. $\square$

Note that $\mathfrak{M}_{g,n}$ is never a DM-algebraic stack, since it contains points with automorphisms group of positive dimension. For example, for a smooth curve $(C, x_1, \dots, x_n) \in \operatorname{Obj}(M_{g,n})(\operatorname{Spec} K)$ with finite automorphisms group, we can construct a nodal curve $\widetilde{C}$, the union of C and $\mathbb{P}^1$ joiniing trasversally $x_n \in C$ with $0 \in \mathbb{P}^1$ and taking section $\widetilde{x}_i := x_i$ for $i < n$ and $\widetilde{x}_n := \infty \in \mathbb{P}^1$. The arithmetic genus of a union of two curves in d points is the sum of the genera plus $d - 1$, so the $\mathrm{p_a}(\widetilde{C}) = g$. Another way to show that the arithmetic genus is the right one is to construct explicitly a family, for example, $\mathfrak{Bl}_{x_n \times \{0\}} C \times \mathbb{A}^1 \to \mathbb{A}^1$, where the general fiber is C and the special is $\widetilde{C}$.

Then $\operatorname{Aut}(\widetilde{C}, \widetilde{x}_i) \cong \operatorname{Aut}(C, x_i) \times \mathbb{G}_m$, since $\varphi \colon \widetilde{C} \xrightarrow{\sim} \widetilde{C}$ with $\varphi(\widetilde{x}_i) = \widetilde{x}_i$ implies $\varphi|_C \colon C \to C \in \operatorname{Aut}(C, x_i)$ and $\varphi|_{\mathbb{P}^1} \in \operatorname{Aut}(\mathbb{P}^1, 0, \infty) \cong \mathbb{G}_m$. This in case $(g, n) \neq (0, 0)$; else we have even more automorphisms.

6.26 definition. A *prestable* genus g curve with n marked points is called *stable* if $\operatorname{Aut}(C, x_1, \ldots, x_n)$ is étale.

6.27 proposition. *Let $(C, x_1, \ldots, x_n)$ a prestable genus g curve. Then the following are equivalent:*

1. *(C, x_i) is stable;*
2. *$\operatorname{Aut}(C, x_i)$ is finite;*
3. *let $\widetilde{C} \to C$ the normalization of C, then for every irreducible component $\widetilde{C}_i$ of $\widetilde{C}$, the numbers $2\operatorname{g}(\widetilde{C}_i) - 2 + n_i$ are positive, where n_i is the number of special points in $\widetilde{C}_i$, that are, points mapped to nodes or to marked points.*

6.28 example. The curve $\widetilde{C}$ obtained attaching a $\mathbb{P}^1$ to a marked point is not stable since it does not satisfy the third condition.

6.29 corollary. *Let (C, x_i) a smooth prestable curve of genus g. Then it is stable if and only if $2g - 2 + n > 0$. This motivates the definition of $M_{g,n}$.*

Proof of Proposition 6.27. The first step is to prove the corollary. If $g \geq 2$, then the automorphisms group of C is finite (with an upper bound of $84(g-1)$); if $g = 1$, then $\operatorname{Aut}(C) \supseteq C$, but fixed some $x \in C$, the automorphisms group of (C, x) is finite; if $g = 0$, $C \cong \mathbb{P}^1$ and if an automorphism fixes three points it is the identity. Conversely, a genus 1 curve without marked points has a dimension 1 automorphisms group (since it contains C itself), and $\mathbb{P}^1$ with only two marked points has infinite automorphisms group.

Consider a curve $(C, x_1, \ldots, x_n)$ with finite automorphisms group; label all special points in $\widetilde{C}$ as $\widetilde{x}_1, \ldots, \widetilde{x}_n, y_1, \ldots, y_m$, where m is the number of nodes in C. Then we have a map $\operatorname{Aut}(\widetilde{C}, \widetilde{x}_i, y_i) \to \operatorname{Aut}(C, x_i)$; we also have an isomorphism of $\operatorname{Aut}(C, x_i)$ with the group $\widetilde{\operatorname{Aut}}$ of automorphisms of $\widetilde{C}$ that fix all $\widetilde{x}_i$ and sends all y_i to y_i but allows exchenges. So we have a short exact sequence

$$1 \to \operatorname{Aut} \to \widetilde{\operatorname{Aut}} \to (\mathbb{Z}/2\mathbb{Z})^m \to 0$$

which says that Aut is finite if and only if $\widetilde{\operatorname{Aut}}$ is finite. In this way, the last two condition are equivalent.

If the last condition is false, than the automorphisms group has positive dimension, and in particular the group is not étale.

The last thing to prove is that the last two conditions implies the first. To prove this we would like to show that the previous exact sequence is not only a group sequence but also a scheme sequence (we wrote $\mathbb{Z}/2\mathbb{Z}$ instead of μ_2 because we want the group with two elements and not the second roots of unity that may cause problems in characteristic 2). We recall that the scheme structure on $\mathrm{Aut}(C, x_i)$ is defined by this property: a map $S \to \mathrm{Aut}(C, x_i)$ is an automorphism $C \times S \to C \times S$ commuting with all sections $x_i \colon S \to C \times S$. The tangent space of $\mathrm{Aut}(C, x_i)$ at id_C is represented by the automorphisms $\varphi \colon C \times \operatorname{Spec} K[\varepsilon]/\varepsilon^2 \to C \times \operatorname{Spec} K[\varepsilon]/\varepsilon^2$ such that if $\varepsilon = 0$ we get the identity. Indeed, thanks to deformation theory, we know that for every scheme X, $\mathrm{T}_{\mathrm{id}}(\mathrm{Aut}(X)) = \mathrm{H}^0(X, \mathrm{T}_X)$. In particular, $\mathrm{T}_{\mathrm{id}}(\mathrm{Aut}(C, x_i)) = \mathrm{H}^0(C, \mathrm{T}_C(-x_1 - \cdots - x_n))$ and $\deg \mathrm{T}_C = 2 - 2g$ and $\deg \mathrm{T}_C(-x_1 - \cdots - x_n) = 2 - 2g - n$. This number is negative by assumptions, so it has no nontrivial section. □

6.30 EXERCISE. Prove that $\mathrm{T}_{\mathrm{id}}(\mathrm{Aut}(C, x_i)) = \mathrm{H}^0(C, \mathrm{T}_C(-x_1 - \cdots - x_n))$.

6.31 REMARK. Given $C \underset{\pi}{\overset{x_i}{\leftrightarrows}} S$ in $\mathfrak{M}_{g,n}(S)$, the set of $s \in S(K)$ such that $(C_s, x_i(s))$ is stable, is open in S.

Proof. This is true since $(C_s, x_i(s))$ is stable if and only if $\mathrm{H}^0(C_s, T_{C_s}(-x_1(s) - \cdots - x_n(s))) = 0$; this is an open condition. □

6.32 DEFINITION. We define the stack $\overline{M}_{g,n}$ in the same way of the stack $\mathfrak{M}_{g,n}$ but adding the condition that the fibers are stable.

6.33 REMARK. Since stable implies prestable, we get a natural morphism $\overline{M}_{g,n} \to \mathfrak{M}_{g,n}$. Moreover, if $2g - 2 + n > 0$, then there is a natural morphism $M_{g,n} \to \overline{M}_{g,n}$.

6.34 EXERCISE. Both these morphisms are open embedding.

6.35 THEOREM (DM for $n = 0$, Mumford and his students for general n).

- *$\overline{M}_{g,n}$ is a smooth DM-algebraic stack of dimension $3g - 3 + n$;*
- *the same for $M_{g,n}$.*

6.36 EXERCISE. Prove that the first statement implies the second using the fact that $M_{g,n} \to \overline{M}_{g,n}$ is an open embedding.

Moreover, $\overline{M}_{g,n}$ is proper over $\operatorname{Spec} K$. This is compatible with the open embedding of $\overline{M}_{g,n}$ into $\mathfrak{M}_{g,n}$ since the latter is highly non-separated.

6.37 REMARK. There exists a natural morphism $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ forgetting the last section. This is the universal curve over $\overline{M}_{g,n}$.

6.38 EXERCISE. Fix $g \geq 2$; let $\alpha \colon \overline{M}_{g,1} \to \overline{M}_g$ be the morphism that forget the section; then this makes $\overline{M}_{g,1}$ into the universal curve over $\overline{M}_g$. In other words, for every $S \to \overline{M}_g$, that is, a family $C \to S$, there exists a natural morphism $C \to \overline{M}_{g,1}$ that makes the diagram cartesian.

6.39 REMARK. This compactification of $M_{g,n}$ is not canonical. There are ongoing works to allow other singularities into the compactification, but to save separatedness one has to remove curves that seemed stable. In higher dimension, the formulation of stableness that extends is the ampleness of $\omega_C \otimes \mathscr{O}_C(\sum x_i)$.

Lecture 11 (2 hours)
February 11th, 2009

6.40 **theorem.** *$\overline{M}_{g,n}$ is an algebraic stack.*

Outline of the proof in case $n = 0$.

1. The first step is to prove that for a stable curve (C, x_i), $\mathscr{L} := \omega_C(\sum x_i)$ is ample and $\mathscr{L}^{\otimes 3}$ is very ample on C; moreover, $\mathrm{h}^1(C, \mathscr{L}^{\otimes 3}) = 0$.

2. If $S \to \overline{M}_{g,n}$ is a morphism, that is, a family $C \underset{\pi}{\overset{\sigma_i}{\leftrightarrows}} S$, ω_π is a line bundle on C such that $s \in S$ implies $\omega_\pi|_{C_s} \cong \omega_{C_s}$; then we have an induced line bundle $\mathscr{L}_S \simeq \omega_\pi \otimes \mathscr{O}_C(\sum \sigma_i(S))$. This is a line bundle since $\sigma_i(S) \subseteq C$ is a Cartier divisor (even this statement is not obvious, and it is due to the fact that the sections are inside the smooth locus). Moreover, we have $\mathscr{O}_C(\sigma_i(S))|_{C_s} \cong \mathscr{O}_{C_s}(\sigma_i(s))$. All these thanks to the fact that π is flat and Gorenstein.

3. We apply cohomology and base change to π and $\mathscr{L}_S^{\otimes 3}$: given a proper and flat morphism, π, and a π-flat sheaf $\mathscr{F} = \mathscr{L}_S^{\otimes 3}$, then for every $i \in \mathbb{N}$ and $s \in S$, there is a natural morphism

$$\varphi^i(s)\colon \mathrm{H}^i(C_s, \mathscr{F}|_{C_s}) \to R^i\pi_\star\mathscr{F}|_{C_s} \otimes_{\mathscr{O}_{S,s}} k(s).$$

This is due to the cartesian diagram

$$\begin{tikzcd} C_s \arrow[r, "\tilde{s}"] \arrow[d, "p"'] & C \arrow[d, "\pi"] \\ \operatorname{Spec} K \arrow[r, "s"'] & S\colon \end{tikzcd}$$

going right and down we get $R^i p_\star \tilde{s}^\star \mathscr{F} = \mathrm{H}^i(S, \mathscr{F}|_{C_s})$, on the other way we get $s^\star R^i\pi_\star\mathscr{F} = R^i\pi_\star\mathscr{F}|_{C_s}$. Moreover:

- if $\varphi^i(s)$ is surjective, then it is an isomorphism, and $\varphi^i(s')$ is also an isomorphism for s' near s;
- if $\varphi^i(s)$ is surjective, then $\varphi^{i-1}(s)$ is an isomorphism if and only if $R^i\pi_\star\mathscr{F}$ is locally free near s.

We can apply this theorem in our case for $i = 2$; since the fiber are 1-dimensional and π is proper, $R^2\pi_\star\mathscr{G} = 0$ for every sheaf $\mathscr{G}$ and in particular $\varphi^2(s)$ is surjective for every $s \in S$. Then by the first consequence it is an isomorphism (but we already know that the cohomology is zero for $i > 1$); but the second consequence tell us that $\varphi^1(s)$ is surjective if and only if $R^2\pi_\star\mathscr{F}$ is locally free near s. Since it is 0, we know that $\varphi^1(s)$ is surjective, so $\mathrm{H}^i(C_s, \mathscr{F}|_{C_s}) = 0$ and $\varphi^0(s)$ is surjective if and only if $R^1\pi_\star\mathscr{F}$ is locally free around s. But we can prove that also $R^1\pi_\star\mathscr{F} = 0$ and so $\varphi^0(s)$ is surjective for all s and $\varphi^{-1}(s)$ is surjective if and only

TODO

if $\pi_\star\mathscr{F} = R^0\pi_\star\mathscr{F}$ is locally free around s. Since $\varphi^{-1}(s)$ is by definition surjective, then $\pi_\star\mathscr{F}$ is locally free around s. At the end, this tell us that $\pi_\star\mathscr{F}$ is locally free and commutes with base change, that is,

$$\pi_\star\mathscr{F} \otimes k(s) \xrightarrow[\text{nat}]{\sim} \mathrm{H}^0(C_s, \mathscr{F}|_{C_s}).$$

4. Let $N \simeq \mathrm{h}^0(C_s, \mathscr{F}|_{C_s})$; to compute N we use Riemann-Roch for nodal curves:

$$N - \mathrm{h}^1(C_s, \mathscr{F}|_{C_s}) = \deg \mathscr{F}|_{C_s} + 1 - \mathrm{g}(C_s).$$

We actually assume for simplicity that there is a smooth fiber; then $\deg \mathscr{F} = 3 \deg \mathscr{L}$, so the right hand side is $3(2g-2+n)+1-g = 5g-5+3n$. Let $P \to S$ be the frame bundle associated to $\pi_\star\mathscr{F}$; it is a principal $\mathrm{GL}(N)$-bundle; call C_P the fiber product $C \times_S P$, then there exists a natural closed embedding $C_P \subseteq P \times \mathbb{P}^{N-1}$ commuting with P. By definition of the Hilbert scheme, this closed embedding defines a map $P \to H := \mathrm{Hilb}^Q(\mathbb{P}^{N-1})$ where Q is some Hilbert polynomial.

5. Replacing P with the projective frame bundle $\tilde{P}$ (a principal $G := \mathbb{P}\,\mathrm{GL}(N)$-bundle over S) we can follow the same argument. Then G acts on $\mathbb{P}^{N-1}$ and hence on H; by definition, there exists a universal family $U_H \to H$ that is flat and lies inside $H \times \mathbb{P}^{N-1} \to H$. Sure not all fibers need to be nodal; let $H^0 \subseteq H$ be the subset of points p_0 such that the fiber over p_0 is connected and nodal; it is an open subset. Thus, $\tilde{P} \to H$ factors set-theoretically throough H^0, but since $H^0 \to H$ is an open embedding the map factors also scheme-theoretically. Moreover, the G actions on H induces an action on H^0 and $\tilde{P} \to H^0$ is G-equivariant.

6. Now, from $\xi \in \overline{M}_{g,n}(S)$, we can construct a principal G-bundle $\tilde{P} \to S$ and a G-equivariant map $\tilde{P} \to H^0$; conversely, from such a diagram, we construct

$$\begin{tikzcd} U_{\tilde{P}} \arrow[r] \arrow[d] & U \arrow[d] \\ \tilde{P} \arrow[r] & H \end{tikzcd}$$

and deduce that $U_{\tilde{P}} \to U$ is G-equivariant. Now, $U_{\tilde{P}} \to \tilde{P}$ is a flat family of genus g nodal connected curves and this family induces a cartesian diagram

$$\begin{tikzcd} U_{\tilde{P}} \arrow[r] \arrow[d] & U_{\tilde{P}/G} \arrow[d] \\ \tilde{P} \arrow[r] & S \end{tikzcd}$$

and so a family over S with the same properties.

7. One could check that this association are one the inverse of the other, so we established an equivalence between $\overline{M}_{g,n}$ and $[H^0/G]$. Last thing to prove is that the stabilizers of G are finite. So we consider a curve $C \subseteq \mathbb{P}^{N-1}$ with $\omega_C^{\otimes 3} \cong \mathcal{O}_{\mathbb{P}^{N-1}}|_C$; we assume this C to correspond to a morphism $\operatorname{Spec} K \to H^0$. Now, G acts on $H^0(K)$, so we consider $\varphi \in G = \operatorname{Aut}(\mathbb{P}^{N-1})$; in particular, $\varphi \colon [C] \to [\varphi(C)]$ and $\varphi \in \operatorname{Stab}_G([C])$ if and only if $\varphi(C) = C$, that is, $\varphi|_C \in \operatorname{Aut}(C)$. We show that $\operatorname{Stab}_G([C]) \to \operatorname{Aut}(C)$ is an isomorphism: assume that $\varphi|_C = \mathrm{id}_C$, then since C is not contained in a hyperplane, it span all $\mathbb{P}^{N-1}$ and $\varphi = \mathrm{id}$; conversely, if $\psi \in \operatorname{Aut}(C)$, it induces an isomorphism $\psi^\star \omega_C \to \omega_C$ that induces an isomorphism of the third powers; then $\mathrm{H}^0(C, \psi^\star \omega_C^{\otimes 3}) \xrightarrow{\sim} \mathrm{H}^0(C, \omega_C^{\otimes 3})$. But global section of a sheaf are the same as global section of its push-forward, and since ψ is an isomorphism, $\psi_\star \psi^\star \omega_C^{\otimes 3} = \omega_C^{\otimes 3}$ and both global sections are isomorphic to $\mathrm{H}^0(C, \omega_C^{\otimes 3}) = \mathrm{H}^0(C, \mathcal{O}_C(1))$; this gives an element of G and remains to check that it restrict to C as ψ. □

6.41 exercise. Compute the Hilbert polynomial Q (since it is constant, it is enough to do it for a smooth fiber).

The case $n > 0$ is just longer to write, not conceptually harder.

As for $\mathfrak{M}_{g,n}$ we just prove that it has smooth charts. We recall that for every $m > n$, we have a natural morphism $\mathfrak{M}_{g,m} \to \mathfrak{M}_{g,n}$ that forgets the last $m - n$ sections. In particular we can compose obtaining $f_m \colon \overline{M}_{g,m} \to \mathfrak{M}_{g,n}$.

6.42 proposition.

1. *The morphisms f_m are representable and smooth;*

2. *the images of f_m for $m > n$ jointly cover $\mathfrak{M}_{g,n}$ (note that finitely many m are not enough);*

3. *composing with the charts for $\overline{M}_{g,n}$ we get smooth charts for $\mathfrak{M}_{g,n}$.*

Proof. We prove first the surjectivity. We consider a connected nodal curve $(C, x_1, \ldots, x_n)$; in general it is not stable, that is, it has some unstable irreducible components $C_1, \ldots, C_k$: we have $a_i := 2gen(\tilde{C}_i) - 2 + N_i \leq 0$, where N_i is the number of special points on $\tilde{C}_i$. Conversely, $a_i \geq -2$, because $\mathrm{g}(\tilde{C}_i)$ and N_i are at least 0. Let $b_i := -a_i + 1$ and $b = \sum b_i$; we choose smooth, distinct, unmarked points $y_1, \ldots, y_b$ such that b_i of them are on C_i. We are marking some additional points to stabilize the unstable components. Then $(C, x_1, \ldots, x_n, y_1, \ldots, y_b)$ is stable and forgetting the last b sections gives the original curve.

We need to prove the smoothness; we will use the formal criterion for smoothness: if $X \to Y$ is a representable (not really necessary) morphism of stack, then it is smooth if for every $S \to Y$ with S a scheme, the base change $S_X \to S$ is smooth. Since we are working with schemes of finite type over an algebraically closed field, smoothness is equivalent to formal smoothness: $S_X \to S$ is formally smooth if for every square zero extension of fat points $T \to \overline{T}$ of $S_X \to S$, there exists a diagonal $\overline{T} \to S_X$. This is actually equivalent if we do not pass through $S_X \to S$, and the proof is easy. Getting back to our case, we just need to prove that $\mathfrak{M}_{g,n+1} \to \mathfrak{M}_{g,n}$ is smooth (since f_m is the composition of an open embedding with it). So we consider a diagram

$$\begin{tikzcd}
\mathfrak{M}_{g,n+1} \arrow[r] & \mathfrak{M}_{g,n} \\
T \arrow[u] \arrow[r] & \overline{T} \arrow[u]
\end{tikzcd};$$

the first vertical map is a family C over T, the second is a family $\overline{C}$ over $\overline{T}$ and we know that π and $\overline{\pi}$ makes the diagram with $\varphi\colon C \to \overline{C}$ and $T \to \overline{T}$ cartesian and that the sections make the diagrams commutative. Now we can use the formal smoothness for

$$\begin{tikzcd}
T \arrow[r, "\varphi\circ\sigma_{n+1}"] \arrow[d] & \overline{C} \arrow[d] \\
\overline{T} \arrow[r, equal] & \overline{T}
\end{tikzcd}.$$

Let $U \subseteq \overline{C}$ the locus where $\overline{C} \to \overline{T}$ is smooth; it is open and $\varphi \circ \sigma_{n+1}$ factors through it. So $U \to \overline{T}$ is smooth and we can find a diagonal map $\overline{T} \to U$. Composing, we get $\overline{\sigma}_{n+1}\colon \overline{T} \to \overline{C}$. We have to check that that $(C, \sigma_1, \dots, \sigma_n, \overline{\sigma}_{n+1})$ is a point in $\mathfrak{M}_{g,n+1}$. Actually, we have to check just the condition on the section $\overline{\sigma}_{n+1}$, since the fibers and π are unchanged. □

Lecture 12 (2 hours)
February 16^{th}, 2009

We have seen that $\overline{M}_{g,n}$ is an open subset of $\mathfrak{M}_{g,n}$, the locus where the automorphisms group is finite (this, as a nontrivial fact, implies that this group is étale in all characteristics).

6.43 LEMMA. *The stack $\overline{M}_{g,n}$ (and so also $\mathfrak{M}_{g,n}$) is smooth of dimension $3g-3+n$.*

Proof. The statement means that there exists an étale cover $\{U_i \to \overline{M}_{g,n}\}$ such that the U_i are smooth schemes of dimension $3g-3+n$. Equivalently, for every $U \to \overline{M}_{g,n}$ étale with U a scheme, U is smooth of dimension $3g-3+n$. Again, equivalently, for every $U \to \overline{M}_{g,n}$ smooth of relative dimension k with U a scheme, U s smooth of relaticeh dimension $3g-3+n+k$. This is a definition, not the way we check the property. Indeed, for every morphism, representable or not, of DM-algebraic stacks, we can use the formal smoothness criterion: $f\colon X \to Y$ is smooth (étale) at $x\colon \operatorname{Spec} K \to X$ if and only if for every 2-commuting diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[d, hook] \arrow[dr] & \\
T \arrow[r] \arrow[d, hook] & X \\
\overline{T} \arrow[r] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}$$

where $T \to \overline{T}$ is a closed embedding of fat points ($T_{\mathrm{red}} = \overline{T}_{\mathrm{red}} = \operatorname{Spec} K$), there exists a φ (with a uniqueness property). Moreover, we may assume that $T \to \overline{T}$ is a square zero extension, or a small extension, that is $\mathscr{I}^2_{T/\overline{T}} = 0$ or $I_{T/\overline{T}} \cdot \mathfrak{m}_{\overline{T}} = 0$. Moreover, $\mathrm{T}_x T$ is the fiber over x of the map

$$X(K[\varepsilon]/\varepsilon^2) \to X(K)$$

induced by $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} K[\varepsilon]/\varepsilon^2$.

Consider $X := \overline{M}_{g,n}$, $Y \simeq \operatorname{Spec} K$. with x corresponding to $(C, p_1, \dots, p_n)$, where C is a stable genus g curve with n marked points. Then we have to give morphisms $T \to X$ and $\overline{T} \to Y$; the second is trivial, the first corresponds to the diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "p_i"'] & T \arrow[d, "s_i"] \\ C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_T \arrow[d] \\ \operatorname{Spec} K \arrow[r] & T. \end{tikzcd}$$

We have to prove existence and uniqueness of a morphism $\overline{T} \to X$, that is, a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "p_i"'] & \overline{T} \arrow[d, "\bar{s}_i"] \\ C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C_{\overline{T}} \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \overline{T}; \end{tikzcd}$$

the condition of commutativity with the other morphisms is precisely an isomorphism $(C_{\overline{T}}|_T, \bar{s}_i|_T) \to (C_T, s_i)$. This is precisely the setup of infinitesimal deformation theory; in particular, if we take the complex analytic approach of Kodaira-Spencer, T and $\overline{T}$ are germs of complex spaces and the deformation is said to be small. The algebraic case was developed by Schlessinger.

Up to now we did not use the extra assumptions of square zero extension or small extension; these will be useful for the deformation theory approach. Let us forget for now the marked points (that is, we consider $n = 0$); we will work with isomorphisms classes. We consider the contravariant functor F from the category of fat points to the category of sets, or, equivalently, the covariant functor from $\mathfrak{Art}$, the category of local finitely generated K-algebra with residue field K to $\mathfrak{Sets}$, defined in this way: $F(A)$ will be the set of cartesian diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_A \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

with $C_A \to \operatorname{Spec} A$ an object in $\overline{M}_{g,0}(\operatorname{Spec} A)$, and we take this set modulo isomorphism that preserve the central fiber, that is, modulo diagrams of the kind

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C_A \arrow[r, "\varphi", "\sim"'] \arrow[d] & C'_A \arrow[d] & C \arrow[l] \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} A \arrow[r, equal] & \operatorname{Spec} A & \operatorname{Spec} K. \arrow[l] \end{tikzcd}$$

If F comes from a DM-algebraic stack then there exist T^1_F, T^2_F, vector spaces of finite dimension over K and for all small extensions

$$0 \to I \to \overline{A} \to A \to 0$$

in $\mathfrak{Art}$, an exact sequence of sets and vector spaces

$$\bullet \to T^1_F \otimes_K I \to F(\overline{A}) \to F(A) \to T^2_F \otimes_K I$$

which is functorial. Here we are interested only in the abelian group structure on the vector spaces, and for exact sequence

$$0 \to G_1 \to S_1 \to S_2 \to G_2$$

we mean:

- exactness at G_1 means that the action of G_1 on S_1 is free;
- exactness at S_1 means that the group G_1 acts transitively on the fibers of $S_1 \to S_2$;
- exactness at S_2 means that an element of S_2 is mapped to $0 \in G_2$ if and only if is in the image of $S_1 \to S_2$.

In our particular case, we start from a family over A and ask ourselves if we can extend it to a family over $\overline{A}$. We can do it if we do not have obstruction in $T^2_F \otimes_K I$; indeed, we call ob_e the map $F(A) \to T^2_F \otimes_K I$, where e is our small extension. Functoriality means that for a diagram

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d] & \overline{A} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & J \arrow[r] & \overline{B} \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

of the two small extensions e and e' (that is, $I \to J$ is K-linear, $\overline{A} \to \overline{B}$ and $A \to B$ are local algebra homomorphisms), then we have an induced diagram

$$\begin{array}{ccccccccc} \bullet \longrightarrow & T^1_F \otimes_K I & \to & F(\overline{A}) & \longrightarrow & F(A) & \to & T^2_F \otimes_K I \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bullet \longrightarrow & T^1_F \otimes_K J & \to & F(\overline{B}) & \longrightarrow & F(B) & \to & T^2_F \otimes_K J \end{array}$$

where the two rightmost squares are commutative and the leftmost is φ-commutative, that is, for every $f \in G_1$ and $s \in S_1$, we have $f(g \cdot s) = \varphi(g) \cdot f(s)$.

Fix a proper scheme C; if as F we consider the same functor as before, but with the additional request of $C_A \to \operatorname{Spec} A$ to be flat, we call F the *deformation functor* of C, Def_C, and in this case $T^i_F = \mathrm{Ext}^i_{\mathscr{O}_C}(L_C, \mathscr{O}_C)$ where L_C is the cotangent complex. If C has lci singularities, then $L_C = \Omega_C$ in degree 0. Now, if C is a nodal projective curve, then $\mathrm{Ext}^2(\Omega_C, \mathscr{O}_C) = 0$ and thanks to the local to global spectral sequence of Ext we have

$$\mathrm{H}^q(\mathcal{E}\mathrm{xt}^p(\Omega_C, \mathscr{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}(\Omega_C, \mathscr{O}_C);$$

in particular, for $p + q = 2$ we have $\mathrm{H}^2(\mathcal{H}\mathrm{om}(\Omega_C, \mathscr{O}_C)) = 0$, $\mathrm{H}^1(\mathcal{E}\mathrm{xt}^1 * \Omega)X, \mathscr{O}_C) = 0$ and $\mathrm{H}^0(\mathcal{E}\mathrm{xt}^2(\Omega_C, \mathscr{O}_C)) = 0$: the first because $\dim C = 1$; the second because $\mathcal{E}\mathrm{xt}^1(\Omega_C, \mathscr{O}_C)$ is supported in $\operatorname{Sing} C$, which has dimension 0; the third because Ω_C has locally a locally free resolution of length 1. □

6.44 exercise. Explicit the uniqueness property of formal étaleness.

6.45 exercise. The fiber is a rigid groupoid because X is a DM-algebraic stack.

6.46 exercise. If X is a DM-algebraic stack and $x \in X(K)$, we can define $\mathrm{T}_x X$ as $\mathrm{T}_{\overline{x}} U$ where

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{\overline{x}} & U \\ & {\scriptstyle x}\searrow & \downarrow{\scriptstyle \pi} \\ & & X, \end{array}$$

π is étale with U a scheme. This is so since for every other diagram with $U' \to X$ and $\overline{x}'\colon \operatorname{Spec} K \to U'$, we have a canonical isomorphism $\mathrm{T}_{\overline{x}'} U' \to \mathrm{T}_{\overline{x}} U$ induced by a canonical isomorphism of both to $\mathrm{T}_{(\overline{x}, \overline{x}')} U \times_X U'$.

6.47 example. Let e be the small extension

$$0 \to K \xrightarrow{\cdot\varepsilon} K[\varepsilon]/\varepsilon^2 \to K \to 0;$$

then the induced exact sequence is

$$0 \to T^1_F \to F(K[\varepsilon]/\varepsilon^2) \to F(K) = \{p\} \xrightarrow{\mathrm{ob}_e} T^2_F;$$

since $F(K)$ is one point, the obstruction map has to be 0 and so we have a natural bijection between $F(K[\varepsilon]/\varepsilon^2)$ and T^1_F. One could see this exact sequence as the one induced by functoriality from a section $K \to K[\varepsilon]/\varepsilon^2$; so since a principal homogeneous space with a marked point is just a vector space, we conclude that the bijection is K-linear.

6.48 REMARK. Up to canonical isomorphism, T^1_F is just the tangent space; conversely, T^2_F is obviously not unique, since we can enlarge it as we want.

6.49 REMARK. If there exists a pair tangent space, obstruction space, with the obstruction space 0, then X is smooth at x.

6.50 EXAMPLE. To prove smoothness and dimensions, usually one uses deformation theory; other cases is when we can, maybe adding structure, represent our moduli stack as a quotient of a scheme: for example, $\overline{M}_{g,n}$ can be constructed as the quotient by $\mathbb{P}\,\mathrm{GL}(N)$ of H_g, the stack with object genus g-curves with a projective basis of $\mathrm{H}^0(C, \omega_C^{\otimes 3})$.

6.51 EXAMPLE. Let $X \subseteq \mathbb{P}^N$ be a projective scheme, $\mathscr{F} \in \mathrm{Coh}(X)$, $P \in \mathbb{Q}[t]$; we can define $\mathrm{Quot}^P_X(\mathscr{F})$ as the fine moduli space of quotient $\mathscr{G}$ of $\mathscr{F}$ with Hilbert polynomial P; in other words, maps $\mathscr{F} \to \mathscr{G}$ such that $\dim \mathscr{G}(n)(X) = P(n)$ for n large enough. We have $\mathrm{Hilb}^P(X) = \mathrm{Quot}^P_X(\mathscr{O}_X)$. In particular, we can study moduli stacks of (semistable) sheaves with Quot-stacks: for example, for the problem of sheaves such that $\mathscr{G}(N_0)$ is generated by global section and $\mathrm{H}^i(X, \mathscr{G}(N_0)) = 0$ for every $i > 0$, then for such a sheaf we have a surjective map $\mathscr{O}_X^{\oplus r} \to \mathscr{G}(N_0)$, or equivalently a surjective map $\mathscr{O}_X(-N_0)^{\oplus r} \to \mathscr{G}$, so we can study this stack starting from a Quot of $\mathscr{O}_X(-N_0)^{\oplus r}$.

6.52 EXAMPLE. Consider a smooth projective variety X and the moduli space $\mathfrak{M}_X$ of coherent sheaves over X. A family over T of coherent sheaves over X is a coherent sheaf $\mathscr{F}$ over $X \times T$ that is T-flat. Thanks to descent theory for coherent sheaves, $\mathfrak{M}_X$ is a stack in the étale topology; moreover, since X is projective, it has open and closed substacks parametrized by the Hilbert polynomial, so we can define its connected components $\mathfrak{M}_{X,P}$. Since all coherent sheaf on X has $\mathbb{G}_m$ inside its automorphisms group, moreover, inside its centre, we can rigidify $\mathfrak{M}$ with respect to $\mathbb{G}_m$, that is, obtain another stack with the same objects but where all automorphisms group are quotiented by $\mathbb{G}_m$. After this rigidification we can get a DM-algebraic stack structure on the open substack of rigid sheaves (with automorphisms group equal to $\mathbb{G}_m$). This is not very satisfactory since not all rigid sheaves are semistable and viceversa.

6.53 REMARK. We have seen a meaning for $\mathrm{Ext}^1(L_C, \mathscr{O}_C)$ and $\mathrm{Ext}^2(L_C, \mathscr{O}_C)$ as the tangent space and the obstruction space. Also $\mathrm{Ext}^0(L_C, \mathscr{O} - C)$ have a simple meaning: it is the tangent space to $\mathrm{Aut}(C)$ at the identity. The meaning of the higher tangent space is something studied by derived algebraic geometry with infinite category.

Lecture 13 (2 hours)
February 17th, 2009

So far, we missed to extend some basic scheme topic to algebraic stacks;

namely, the issues about separated or proper stacks, and (quasi)-coherent sheaves on stacks.

Recall that a scheme is said to be separated if the diagonal morphism is a locally closed embedding. If X is a DM-algebraic stack, then Δ_X is a representable morphism; for every geometric point $x\colon \operatorname{Spec} K \to X$ (i.e., $x \in \operatorname{Obj}(X(K))$), we have a cartesian diagram

$$\begin{tikzcd} \operatorname{Aut}(x) \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d, "x"] \\ \operatorname{Spec} K \arrow[r, "x"'] & X. \end{tikzcd}$$

Moreover, we observe that id $S \times_X T$ is the fiber product of the two maps f and g, then it is the fiber product also of the maps $\Delta_x\colon X \to X \times X$ and $(f,g)\colon S \times T \to X \times X$. So we have a cartesian diagram

$$\begin{tikzcd} \operatorname{Aut}(x) \arrow[r] \arrow[d] & \operatorname{Spec} K \arrow[d, "(x{,}x)"] \\ X \arrow[r, "\Delta_X"'] & X \times X. \end{tikzcd}$$

At this time, note that if Δ_X is a closed embedding, then it has to be at least injective on points, so one has $\operatorname{Aut}(x) = \{\mathrm{id}_x\}$. This tell us that we cannot use the same property to define separated algebraic stacks.

6.54 definition. A DM-algebraic stack X is *separated* if Δ_X is proper

6.55 remark. A locally closed embedding is proper if and only if is a closed embedding; hence this definition extends the usual definition of separatedness for schemes.

6.56 definition. A morphism of DM-algebraic stacks $f\colon X \to Y$ is *separated* if $\Delta_f\colon X \to X \times_Y X$ is proper.

6.57 proposition (Valuative criterion of separatedness). *A morphism of DM-algebraic stacks $f\colon X \to Y$ is separated if and only if for every smooth irreducible curve C, for every open non-empty $U \subseteq C$, for every 2-commutative diagram*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "\overline{g}"'] & Y \end{tikzcd}$$

with 2-morphism η, for every two factorizations $h, \overline{h}\colon C \to Y$ with 2-morphisms $\alpha\colon h \circ i \Rightarrow g$, $\beta\colon \overline{g} \Rightarrow f \circ h$, $\overline{\alpha}\colon \overline{h} \circ i \Rightarrow g$, $\overline{\beta}\colon \overline{g} \Rightarrow f \circ \overline{h}$ such that $\eta = (\mathrm{id}_f \circ \alpha) \circ (\beta \circ \mathrm{id}_i) = (\mathrm{id}_f \circ \overline{\alpha}) \circ (\overline{\beta} \circ \mathrm{id}_i)$, then there exists a unique 2-isomorphism $\delta\colon h \Rightarrow \overline{\delta}$

such that $\alpha = \overline{\alpha} \circ (\delta \circ \mathrm{id}_i)$, $\overline{\beta} = (\mathrm{id}_f \circ \delta) \circ \beta$.

6.58 EXAMPLE. We can use the valuative criterion to prove that M_g is separated (over $\operatorname{Spec} K$) for $g > 0$. Consider a smooth irreducible curve V and an open non-empty $U \subseteq V$ with a map $U \to M_g$. This corresponds to a proper smooth morphism $p \colon C_u \to U$ with genus g curves as fibers. Two factorizations corrisponds to two families $\pi \colon C_V \to V$ and $\overline{\pi} \colon \overline{C}_V \to C$ with the same properties, such that the induces the same family π over U. Then we have to prove that there exists a unique isomorphism $C_V \to \overline{C}_V$ such that $\overline{h} = \delta \circ h$, where $h \colon C_U \to C_V$ and $\overline{h} \colon C_U \to \overline{C}_V$. So we reduce to a problem on the smooth surfaces C_V and $\overline{C}_V$. Since we have an isomorphism between $\pi^{-1}(U)$ and $\overline{\pi}^{-1}(U)$, then the two surfaces C_V and $\overline{C}_V$ are birational; in particular, we can find surfaces S and $\overline{S}$, obtained by blowing up C_V and $\overline{C}_V$ with an isomorphism between them; moreover, we assume that they are minimal, in the sense that the exceptional curve of the last blow up is not contracted by the isomorphism (in each direction). Since everything is separated, then all these maps commutes with the projection π and $\overline{\pi}$. Consider the last exceptional curve E in S; since it is contracted to a point in C_V, it is also a point in V; so E_1 is mapped to a fiber of $\overline{\pi}$ into $\overline{C}_V$; but a fiber is a smooth genus g curve and so E_1 is mapped to a point. Then there cannot be any blow-ups and C_V is isomorphic to $\overline{C}_V$, unique since it is determined on a open dense set.

There are other version of the valuative criterion:

1. if we do not want to work with a ground field, we have to consider $C = \operatorname{Spec} R$ where R is a (discrete) valuation ring and $U \subseteq C$ is an open point; discrete is enough when working with locally Noetherian schemes;

2. in the complex category, C is a disk in $\mathbb{C}$ and U is the punctured disk;

3. the formal version, for schemes of a finite type over an algebraically closed field: $C = \operatorname{Spec} K[[t]]$ and $U = \operatorname{Spec} K((t))$ (the Laurant series);

4. in each versions, we can fix a priori an open dense substack $X' \subseteq X$ (dense means that for all $q \colon U \to X$ with U a scheme and q étale or smooth morphism, $q^{-1}(X')$ is dense) and assume that g factors through X'; note that this last property holds also for schemes.

There is a definition of properness for a morphism of stacks; we skip it to go directly to the valuative criterion.

6.59 PROPOSITION (Valuative criterion for properness). *With the same assumptions of the previous criterion, $X \to Y$ is proper if there exists a factorization, possibly after a finite base change on C (that is, a finite map $\overline{C} \to C$).*

We can explain the finite base change for example in the holomorphic case: $\overline{C}$ is again a disk, but the map is z^r for $r > 0$.

6.60 EXERCISE. In the usual setup, if $U = C \setminus \{p\}$, then we can restrict C and consequently U as long the restriction contains p.

In the usual setup, since we may restrict C, we may assume $\mathscr{O}_C(p)$ to be trivial, and $\overline{C} \to C$ can be the degree r cover if C branched on p

6.61 EXAMPLE. Let $g \geq 2$; then $\overline{M}_g$ is proper. To prove this, we use the last version, that is, we consider g to factor through an open dense substack.

So we consider $U \hookrightarrow V$ where V is a smooth irreducible curve, U is an open and non-empty subset (we assume also that $U = V \setminus \{p\}$, there is no difference), $X := \overline{M}_g$, $Y := \operatorname{Spec} K$, $X' := M_g$. We have to prove that X' is dense in X; this comes from deformation theory: every étale morphism $T \to X$ with T a scheme, corresponding to a family $C_T \to T$, with a fiber $C_t \mapsto t$ singular, then we have

$$\mathrm{T}_t T \xrightarrow{\sim} \operatorname{Ext}^1(\Omega_{C_t}, \mathscr{O}_{C_t}) \xrightarrow{\text{su}} \mathrm{H}^0(\mathcal{E}\mathrm{xt}^1(\Omega_{C_t}, \mathscr{O}_{C_t})) \to \mathrm{H}^2(\mathcal{H}\mathrm{om}(\Omega_{C_t}, \mathscr{O}_{C_t})) = 0.$$

TODO

The morphism $g\colon U \to M_g$ corresponds to a family $\pi\colon C_U \to U$ and we want to extend it to a family $C_V \to V$ possibly with nodal fibers. One possible approach is to assume, after restricting U and V, that $\pi_\star \omega_{C_U/U}^{\otimes 3}$ gives a projective embedding into $\mathbb{P}^N \times U$ and we take closure inside $\mathbb{P}^N \times V$. We call this family $C'_V \to V$; it is possibly very singular. We can resolve the singularities with a birational proper morphism $\widetilde{C}_V \to C'_V$ with $\widetilde{C}_V$ smooth. Since it is an isomorphism over the smooth locus, it is birational to C_U. With the embedding resolution of singularities, we may assume that $(\widetilde{C}_{V,p})_{\mathrm{red}}$ is at most nodal. Thanks to the finite base change, modulo normalization, we can get rid of the non-reduced issues, obtaining some family $S \to \overline{V}$.

In such a family, the central fiber can be non-stable if and only if it has genus 0 components meeting the rest of the curve in 1 or 2 points (it is connected because the family is flat and the other fibers are connected). We consider $N_{C_i/S}$, where C_i is some destabilizing component with i point of contact with the rest of the curve, then $C_p \cdot C_i = 0$, but $C_{v_0} = C_i + R_i$ and so $C_i^2 = -i$ and we can contract these curve to obtain the relative canonical model, that is essentially unique. In all these stuff, we used the X' thing to assure that outside p we did not change anything.

6.62 EXERCISE. If C is a nodal curve, then $\mathcal{E}\mathrm{xt}^1(\Omega_C, \mathscr{O}_C) \cong \bigoplus_{p \in \operatorname{Sing}(C)} \mathscr{O}_p$. Hint: the statement is étale local, so we can consider $C = \operatorname{Spec} K[x,y]/(xy)$ and we have to find an explicit free resolution of Ω_C as a $(K[x,y]/(xy))$-module.

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