

MINIMAL MODEL PROGRAM: THEORY AND APPLICATIONS

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1 INTRODUCTION TO MMP

Let X, Y be projective varieties of dimension n ; we say that X and Y are *birational* if there are Zariski open subschemes $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $X_0 \cong Y_0$.

In dimension 1, birationality is equivalent to isomorphism. In dimension 2, we have non-trivial birationality relations; for example, if we blow up a point in a surface, then the surface we obtain is birational to the one we started with, but they are not isomorphic.

The problem that the mmp tries to attack is, given a normal, projective variety X , to find a *minimal model*, that is an X_{\min} birational to X such that X_{\min} is as simple as possible.

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1.1 *Intuitive approach*

A first tentative to solve this problem is the following: consider a partial order in the set of varieties: $X \geq Y$ if and only if there exists a birational morphism $X \rightarrow Y$; then we define X to be minimal if $X \geq Y$ implies $X \cong Y$.

This tentative definition has some problems.

1. Does the minimal model exist? Given X , how do we get X_{\min} ?
2. Does this really give us a “simple” variety?
3. Do we have a criterion to know if X is minimal?

To explain the second problem, consider $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$; this is a \mathbb{P}^1 -bundle over \mathbb{P}^1 , with a (-2) -curve. This has a natural contraction morphism to a quadric cone in \mathbb{P}^3 , and this morphism is birational. Even if the quadric cone is not a complicated object, this shows that birational morphisms can go to variety that tends to go outside the category we are interested in, namely algebraic varieties. Hence, this highlight that we have to limit the kind of birational morphisms we use.

1.2 *Classical approach*

This approach is the classical one, developed by the Italian school and Castelnuovo in particular. More precisely, he gave the following.

1.1 DEFINITION. A smooth projective surface X is *minimal* if there are no (-1) -curves in X .

This is a good criterion, also because it comes with an algorithm.

1.2 THEOREM. *Let X be a smooth projective surface; if it contains a (-1) -curve C , then there exists a smooth projective surface Y and a point $p \in Y$ such that $X = \text{Bl}_p Y$ and C is the preimage of p via the blowing up morphism.*

This theorem gives us the following algorithm:

1. start with a smooth projective surface X ;
2. if X does not contain a (-1) -curve, X is minimal;
3. if X contains a (-1) curve, contracts it and substitute X with the new surface;
4. go back to the starting point.

And the algorithm has an end because a blow up increase the Néron-Severi number by 1, so the number of contraction we can do is bounded by the Néron-Severi number of X .

Thanks to the following, it turns out that this approach is equivalent to the intuitive one we explained before.

1.3 LEMMA. Let X be a smooth projective surface without (-1) -curves, and let $\mu: X \rightarrow Y$ be a birational morphism with Y smooth and projective. Then μ is an isomorphism.

Proof. If μ is finite, then it is an isomorphism (result of Zariski). Hence, suppose that μ is not finite; then it contracts curves C_1, \dots, C_γ . Suppose that it contracts just one curve C to a point $p \in Y$. Let $Y' := \text{Bl}_p(Y)$; the preimage of p in Y' is a (-1) -curve E , and by the universal property of the blow up, we have a morphism $X \rightarrow Y'$; it is easy to prove that this is indeed an isomorphism so $C^2 = E^2 = -1$. \square

Of course, by the example of the quadric cone we gave before, this lemma fails when Y is singular.

1.3 Mmp approach

If $L \rightarrow X$ is a Cartier divisor, then L is said to be *nef* if $L \cdot C \geq 0$ for every curve C . If X is a smooth projective surface, and $C \subseteq X$ is a (-1) -curve, then $K_X \cdot C = -1$, so K_X is not nef. In other words, the nefness of K_X is enough to show that X is minimal in the sense of Castelnuovo.

This works in any dimension, as the following shows.

1.4 LEMMA. If X is smooth and projective, with K_X nef, and $\mu: X \rightarrow Y$ is a birational morphism with Y smooth and projective, then μ is an isomorphism.

The proof is essentially the same as before. Moreover, this lemma lead us to the following.

1.5 DEFINITION. Let X be a smooth projective variety; then X is a *minimal model* if K_X is nef.

1.6 REMARK. This definition differs from Castelnuovo also in the surface case; for example, the Hirzebruch surfaces have no minimal models in this sense, because each smooth surface birational to them has K_X which is not nef. The solution to the problem of not having a minimal model for some variety is solved using the following.

1.7 DEFINITION. We say that X is a *Mori fiber space* if there exists $\varphi: X \rightarrow Y$ such that $\dim X > \dim Y$ and $K_X|_{X_y}$ is antiample for each $y \in Y$.

The aim of the mmp is then show that each variety has a minimal model, or is a Mori fiber space. This is the algorithm we obtain, modulo proving a lot of requirements:

1. start with X ;
2. if K_X is nef, then X is minimal and we finish;
3. if K_X is not nef, then there exists a contraction $\varphi: X \rightarrow Y$ such that K_X restricted to the fibers of φ is antiample;

4. if $\dim X > \dim Y$, then X is a Mori fiber space and we finish;
5. if $\dim X = \dim Y$, then we replace X by Y and we start again.

1.8 EXAMPLE. Let $X := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2))$; this is a \mathbb{P}^1 -bundle over \mathbb{P}^2 with a K_X -negative divisor E (i.e., $K_X|_E \cong \mathcal{O}_{\mathbb{P}^2}(-1)$); we can contract E with a birational morphism $\mu: X \rightarrow Y$ that contracts E to a point p . This shows that the algorithm is not closed for the class of smooth and projective varieties.

This is a problem at least because to run the algorithm we need a variety where the nefness of K_X makes sense, i.e., we need at least that K_X is \mathbb{Q} -Cartier. Hence, we need to modify the algorithm requiring at each step that K_X is \mathbb{Q} -Cartier.

1.9 EXAMPLE. Let $X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2})$; this is a \mathbb{P}^2 -bundle over \mathbb{P}^2 , with a surface S that can be contracted to a point p via a birational morphism $\mu: X \rightarrow Y$. The problem is that K_Y is not \mathbb{Q} -Cartier.

The idea is that to repair the problem created by the contraction morphism using a *flip*: since we cannot continue with Y , we construct a variety X^+ and a morphism $q^+: X^+ \rightarrow Y$ with \mathbb{Q} -Cartier K_{X^+} and K_{X^+} ample on every fiber. So, we modify again the algorithm: before replacing X by Y , we need to ensure that K_Y is \mathbb{Q} -Cartier; if it is not, then we have to make a flip and we replace X by X^+ .

The real problem now is that we have to prove that all the requirements we had in the diagrams holds, in particular the most important is the existence of flips. This is in general not true for the class of varieties with K nef.

1.10 DEFINITION. Let X be a normal variety with K_X \mathbb{Q} -Cartier; X has *terminal singularities* if for every birational morphism $\mu: Y \rightarrow X$, we have

$$(1) \quad K_Y = \mu^*K_X + \sum a_i E_i$$

with $a_i > 0$, where $E_i \subseteq Y$ are the exceptional divisor.

This class is indeed stable under the mmp and contains the smooth varieties. Moreover, it is the minimal class with these properties.

2 SINGULARITIES OF PAIRS

2.1 Why discrepancies

The discrepancies are the coefficients a_i in Equation (1). They are a measure of the singularities. Let us show this through some observations.

1. If X is a normal surface, then there exists a unique minimal resolution $\mu: X' \rightarrow X$ with X' smooth; moreover, $K_{X'} = \mu^*K_X + E$, with $E^2 \leq 0$ (here we don't assume K_X to be \mathbb{Q} -Cartier because on a surface we can always pullback in the sense of Mumford). If $E = 0$, then we have the ADE singularities (rational double points). If $E = \sum a_i E_i$ with $a_i > -1$,

then X has quotient singularities. The moral is that low discrepancies means hard singularities.

2. If X' is smooth and projective, $\mu: X' \rightarrow X$ is a birational contraction of E to a point, such that $K_{X'}|_X$ is antiample. If moreover K_X is \mathbb{Q} -Cartier, then we can write $K_{X'} = \mu^*K_X + aE$. If $C \subseteq E$ is a curve, then $E \cdot C < 0$ and $K_{X'} \cdot C < 0$ and this implies $a \geq 0$ (i.e., X has at most terminal singularities).

2.2 Log pairs

2.1 DEFINITION. Let X be normal, $D = \sum a_i D_i$ be a \mathbb{Q} -linear combination of integral Weil divisor. We say that D is \mathbb{Q} -Cartier if mD is Cartier for $m \in \mathbb{N}$.

2.2 DEFINITION. A pair (X, Δ) is a *log pair* if X is a normal variety, $K_X + \Delta$ is \mathbb{Q} -Cartier and $\Delta = \sum d_i D_i$ with $d_i \in [0, 1]$.

The philosophy is that in some situation, the canonical divisor is not \mathbb{Q} -Cartier, but we can deform it slightly in order to get a \mathbb{Q} -Cartier divisor.

Let now (X, Δ) be a log pair, and $\mu: Y \rightarrow X$ be a birational morphism; then we write

$$K_Y = \mu^*(K_X + \Delta) + \sum_{E \subseteq Y \text{ prime}} a_E(X, \Delta)E.$$

We call $a_E(X, \Delta)$ the *discrepancy* of E with respect to (X, Δ) . Up to now these coefficients are not really well defined; so we make a convention: in the sum there are two cases: if E is contracted by μ , there is already no ambiguity; if $\mu(E)$ is a divisor, then $a_E(X, \Delta) \neq 0$ if and only if $\mu(E) = D_i$, and in this case $a_E(X, \Delta) = d_i$.

One could write the same formula without assuming this convention, extracting from the sum the strict transform of the D_i .

2.3 REMARK. One could ask why we define things using discrepancy greater or greater or equal than 0 or -1 , and not with other limits. The reason is the following: if $E \subseteq Y$ is such that $a_E(X, \Delta) < -1$, then there exists $\mu': Y' \rightarrow X$ with E' of arbitrarily small discrepancy. For example, if $a_E(X, \Delta) = -1 - c$, we can take $Z_0 \subseteq E$ of codimension 1, blow it up, obtaining an exceptional divisor E_0 of discrepancy $-c$, and blowing up the intersection of E and E_0 we get a divisor of discrepancy $-2c$. In this way we get all the multiples of $-c$.

2.4 DEFINITION. A log pair (X, Δ) is *log canonical (lc)* if $a_E(X, \Delta) \geq -1$; is *Kawamata log terminal (klt)* if $a_E(X, \Delta) > -1$ (for all $E \subseteq Y$ prime divisor and all $\mu: Y \rightarrow X$).

Note that since a pair has $d_i \in [0, 1]$, then a klt pair (by our definition) has $d_i < 1$.

2.5 EXAMPLE. We will show that being lc sometimes is a numerical condition, but this is not the case in general. Let X be smooth, $\Delta \subseteq X$ an effective divisor.

1. If $\text{mult}_x \Delta \geq n \simeq \dim X$ then (X, Δ) is not lc: let $\mu: Y := \text{Bl}_x X \rightarrow X$, then $K_Y = \mu^* K_X + (n-1)E$, then $\mu^* \Delta = \Delta' + \text{mult}_x \Delta E$ and this implies that $a_E(X, \Delta) = n-1 - \text{mult}_x \Delta < -1$.
2. If X is a surface, divisor with a node (reducible or irreducible) are lc, but a cusp is not lc (even if it has multiplicity less or equal than 2 for all points). To see this, blow up three times the cusp point, then we have $K_{X'} = \mu^* K_X + E_1 + 2E_2 + 4E_3$, while $\mu^* \Delta = C + 2E_1 + 3E_2 + 6E_3$, and we have $a_{E_3}(X, \Delta) = -2$ (one stops the blowing up process then the divisors intersect transversally and are smooth).

2.6 DEFINITION. Let (X, Δ) be a log canonical pair. We say that a subvariety $\omega \subseteq X$ is a *lc center* if there exists $\mu: Y \rightarrow X$ birational, with $E \subseteq Y$ mapping to ω such that $a_E(X, \Delta) = -1$.

We have the following generalization of Kodaira's vanishing theorem.

2.7 THEOREM (Nadel). *Let (X, Δ) be a lc pair with X projective, M a Cartier divisor such that $M - (K_X + \Delta)$ is nef and big, then $H^i(X, M \otimes \mathcal{I}_\omega) = 0$ for every $i > 0$ where ω is the union of the lc centers of (X, Δ) .*

In particular, if we look at the sequence

$$0 \rightarrow \mathcal{I}_\omega \otimes M \rightarrow M \rightarrow M|_\omega \rightarrow 0,$$

then Nadel vanishing says that $H^0(X, M)$ surjects onto $H^0(M, M|_\omega)$. In other words, Nadel's theorem gives an induction strategy on the dimension. The problem is that the theorem does not tell what is the geometry of ω .

2.3 Problems in understanding the lc centers

2.8 EXAMPLE. Let X be a smooth surface, $\Delta = \sum d_i D_i$, with the D_i smooth and intersecting transversally. In this case, (X, Δ) is lc. Suppose that $d_0 = 1$, then D_0 is a lc center. If also D_1 meets D_0 in a point p and $d_1 = 1$, then p is a lc center too. There are two situations now: there exists a 0-dimensional center, or $d_i < 1$ for all i such that $D_i \cap D_0 \neq \emptyset$. In the latter case, $(D_0, \sum d_i D_i \cap D_0)$ is a klt pair. Moreover, by subadjunction,

$$(K_X + \Delta)|_{D_0} \cong K_{D_0} + \sum d_i D_i \cap D_0.$$

2.9 THEOREM (Kawamata). *Let (X, Δ) be a lc pair, and $\omega \in X$ a minimal lc center. Let $H \subseteq X$ be an ample divisor; then for every $\varepsilon > 0$, there exists a divisor $\Delta_\omega \subseteq \omega$ such that (ω, Δ_ω) is klt and $K_\omega + \Delta_\omega = (K_X + \Delta + \varepsilon H)|_\omega$.*

In other words, if we add just a little bit of positivity, we can do an adjunction formula.

2.4 An application

2.10 THEOREM. Let X be a smooth projective surface such that $K_X \cong \mathcal{O}_X$; let $A \subseteq X$ be an ample Cartier divisor, $D \in |A|$ a general element. Then (X, D) is lc; in particular D is reduced.

2.11 REMARK. By Bertini's theorem, we know that the singular locus of D is contained in $\text{Bs } |A|$.

Proof. If (X, D) is not lc, we can make it so by lowering the coefficients of the boundary divisor. That is, (X, cD) is lc (but not klt) for some $0 < c < 1$. In particular, there exists a log canonical center $\omega \subseteq X$ for (X, cD) , and with some perturbation technique, we can assume that there exists only one. Note that ω is contained in $\text{Bs } |A|$. Now,

$$A - \underbrace{(K_X + cD)}_{=cA} = (1 - c)A$$

is ample. Applying Nadel's theorem, we have $H^0(X, A) \rightarrow H^0(\omega, A|_\omega)$. If ω is a point, we already have a contradiction. So suppose it is a curve. Then, by Kawamata, (ω, Δ_ω) is klt and therefore ω is a smooth curve. By subadjunction,

$$A|_\omega - (K_\omega + \Delta_\omega) = A|_\omega - (K_X + \Delta + \varepsilon A) - (1 - c - \varepsilon)A.$$

This in particular implies $\deg A|_\omega > \deg K_\omega$, and for a curve this is enough to ensure that $A|_\omega$ has a global section, i.e. $H^0(\omega, A|_\omega) > 0$. Hence, A has a section and this contradicts the fact that ω was in $\text{Bs } |A|$. \square

3 FANO MANIFOLDS AND HODGE THEORY

In the following we are going to see some applications of Theorem 2.10.

Lecture 3 (1 hour)
October 7th, 2010

3.1 EXERCISE. Theorem 2.10 holds even if X has canonical singularities.

3.2 THEOREM. Let (X, Δ) be a log pair, $S \subseteq X$ an integral divisor, $S \not\subseteq \text{Supp } \Delta$. Then $(X, S + \Delta)$ is lc near S if and only if $(S, \Delta|_S)$ is lc.

3.3 REMARK. The direction \Rightarrow is easy, the surprising part is the other direction.

3.1 Anticanonical divisors on Fano manifolds

Recall that by definition, X is Fano if and only if $-K_X$ is ample.

3.4 EXAMPLE. The projective spaces are Fano, $\text{Bl}_{p_1, \dots, p_n} \mathbb{P}^2$ is Fano for $n \leq 8$ when the points are in general position.

We have a complete classification of Fano varieties of dimension less or equal than 3. The case of $\rho(X) = 1$ cannot be treated with the mmp, and it was accomplished by the Russian school. Actually, they did the classification

in the hypothesis that there is a smooth anticanonical divisor, and some years later Shokurov proved that this is true.

3.5 THEOREM (Shokurov, 1980). *Let X be a smooth Fano threefold; then a general element in $|-K_X|$ is a smooth surface.*

3.6 REMARK. By Kodaira vanishing, $h^i(X, -K_X) = 0$ for $i > 0$; also, by Riemann-Roch we have

$$h^0(X, -K_X) = \chi(X, -K_X) = 2.$$

In particular, if $S \in |-K_X|$ is a smooth surface, then $K_S \cong \mathcal{O}_S$ and $h^1(S, \mathcal{O}_S) = 1$, so S is a K3 surface.

Idea of the proof of Theorem 3.5. The modern proof of Shokurov's theorem is divided in two steps:

1. if $S \in |-K_X|$ is general, then it has canonical singularities;
2. use the classification of linear systems on K3 surfaces (Saint-Donat, 1975).

By inversion of adjunction, the first step is equivalent to proving that (X, S) is lc, and that all the lc centers are divisors (this condition is summed up saying that (X, S) is *plt*).

We will give here a different proof of second part. Fix $S_1 \in |-K_X|$ general. By the first step, S_1 has canonical singularities, hence by adjunction $K_{S_1} \cong \mathcal{O}_{S_1}$. Consider another $S_2 \in |-K_X|$ general, and let $C := S_1 \cap S_2 \in |-K_X|_{S_1}$. This is again a general element of that linear system. Then, by Theorem 2.10, (S_1, C) is lc, and C is reduced.

Therefore, the $\text{Sing } C$ is a finite set, and being C a complete intersection of two general divisors, we have $\text{Sing } S_1 \cup \text{Sing } S_2 \subseteq \text{Sing } C$.

By replacing S_2 by a sufficiently general member of the pencil generated by S_1 and S_2 , we can assume that the singular points of C are singular both on S_1 and on S_2 . So let p be a singular point of S_1 and S_2 ; we obtain $\text{mult}_p(S_1 + S_2) = 4 \geq \dim(S_1 + S_2)$, therefore, $(X, S_1 + S_2)$ is not lc. But, inversion of adjunction implies that $(S_1, S_2|_{S_1} = C)$ is not lc, and this is a contradiction. \square

In the following, we restrict to dimension at most 4.

3.7 THEOREM (Kawamata (2000)). *If X is a Fano fourfold, and $Y \in |-K_X|$ is general, then Y has canonical singularities.*

3.8 THEOREM (Höring-Voisin). *If X is a smooth Fano fourfold, and $Y \in |-K_X|$ is general, then Y has isolated singularities.*

3.9 REMARK. The proof of this theorem is similar to the new one we saw for Shokurov's theorem. Also, the result is the best obtainable. Consider $S := \text{Bl}_{p_1, \dots, p_8} \mathbb{P}^2$; then $\text{Bs } |-K_S| = |p_i|$; consider $X := S \times S$, then $\text{Bs } |-K_X| = S_1 \cup S_2$, where $S_i := \pi_i^{-1}(\{p_i\})$ and π_i are the projections $X \rightarrow S$. If $Y \in |-K_X|$ is general, then $S_1 \cup S_2 \subseteq Y$, and S_i are Weil divisor that are not Cartier (indeed,

if they were, then $\dim S_1 \cap S_2 \geq 1$, but their intersection is just $\{p_i\}$. So X is not \mathbb{Q} -factorial, hence cannot be smooth.

3.2 Hodge conjecture with integral coefficients

Let X be a complex projective manifold. Then the Hodge decomposition is

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X),$$

where on the left we have de Rham cohomology and on the right we have Dolbeaut cohomology.

If $Z \subseteq X$ is a subvariety of codimension k , then $[Z] \in H^{2k}(X, \mathbb{C})$ and we know that $[Z] \in H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z})$. We say that α is a Hodge class if this condition holds, i.e. if $\alpha \in H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z})$.

3.10 CONJECTURE (Hodge). *The class α is a Hodge class if and only if $\alpha = \sum \alpha_i Z_i$ with $Z_i \subseteq X$ subvariety and $\alpha_i \in \mathbb{Q}$.*

The Hodge conjecture is false if we impose $\alpha_i \in \mathbb{Z}$. A counterexample has been given by Kollár: let $X^3 \subseteq \mathbb{P}^4$ be a very general hypersurface of degree 7^3 . Then $H^4(X, \mathbb{Z}) = \mathbb{Z}\alpha$, and α is not algebraic.

3.11 THEOREM (Voisin). *Let X be a smooth projective Calabi-Yau threefold, with $K_X \cong \mathcal{O}_X$, $h^1(X, \mathcal{O}_X) = h^2(X, \mathcal{O}_X) = 0$. Then the integral Hodge conjecture holds for curves.*

3.12 THEOREM (Höering-Voisin). *If X is a smooth Fano fourfold, then the integral Hodge conjecture holds for curves.*

Using these Hodge classes of curves, we can define a birational invariant of a variety (Soulé-Voisin). If X is rational, then integral Hodge conjecture holds; therefore, if it fails we know that X is not rational.

To prove the theorem, the obvious idea is to consider $Y \in |-K_X|$ a smooth divisor; then Y is a Calabi-Yau threefold, and by Lefschetz one has $H^4(Y, \mathbb{Z}) \rightarrow H^6(X, \mathbb{Z})$. The problem is that we don't have smoothness but only isolated singularities.

To solve the problem, one look at the proof of Theorem 3.11. One take a very ample divisor $H \subseteq Y$; then by Lefschetz $H^2(H, \mathbb{Z}) \rightarrow H^4(Y, \mathbb{Z})$. In our situation, Y is not smooth, but having isolated singularities, $H \subseteq Y$ very ample is a smooth surface contained in the smooth locus of Y , and one can apply Lefschetz again.

3.13 THEOREM (Voisin). *There exists a \mathbb{Z} -bases $(\alpha_1, \dots, \alpha_n)$ of $H^2(H, \mathbb{Z})$ such that α_i becomes a Hodge class in a small deformation.*