

# VIRTUAL PULLBACKS ON ALGEBRAIC STACKS

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## CONTENTS

1	Reviews	1
1.1	Intersection theory . . . . .	1
1.2	Algebraic stacks . . . . .	3
1.3	Key example . . . . .	6
2	Mixing	7
2.1	Intersection theory for algebraic stacks . . . . .	7
2.2	Algebraic stacks for intersection theory . . . . .	7
2.3	How does one apply this? . . . . .	10

## 1 REVIEWS

The idea of this lecture is to mix two area of algebraic geometry, namely intersection theory and algebraic stacks, and produce new results. We start with a review of these areas.

*Lecture 1 (1 hour)  
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### 1.1 Intersection theory

When we say *scheme*, we mean a scheme of finite type over a field  $K$ , which most of the time is thought to be algebraically closed, but also most of the time this request will not be necessary.

So let  $X$  be a scheme,  $V \subseteq X$  a subvariety, i.e. a closed, irreducible reduced subscheme of dimension  $d$ . We define  $Z_d(X)$  to be the free abelian group generated by the classes subvarieties  $V$  of dimension  $d$ .

If  $Y \subseteq X$  is a closed, pure  $d$ -dimensional subscheme, then  $[Y] = \sum m_i [Y_i]$ , where  $m_i$  is the multiplicity of  $Y_i$  along  $Y$ .

We have some functoriality properties:

1. if  $f: X \rightarrow Y$  is a proper morphism, then there exists a map  $f_*: Z_d(X) \rightarrow Z_d(Y)$  called *proper pushforward*;

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## 1. REVIEWS

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2. if  $f: X \rightarrow Y$  is a flat map of relative dimension  $r$ , then there exists a map  $f^*: Z_d(Y) \rightarrow Z_{d+r}(X)$  that sends  $[V]$  to  $[f^{-1}V]$ , called *flat pullback*.

We would like to do with these groups something like cohomology; indeed, usually one works not with cycle groups but with cycle classes groups, there the equivalence relation is rational equivalence. To see why this is needed, we propose the following example.

**1.1 EXAMPLE.** Let  $X$  be a scheme,  $L \in \text{Pic}(X)$ ; then we would like to define a map  $c_1(L): Z_{d+1}(X) \rightarrow Z_d(X)$ . So, let  $W$  be a subvariety of dimension  $d+1$ , and  $s$  be a rational section of  $L|_W$ ; then we can define  $c_1(L) \cap [W]$  by  $s_0 - s_\infty$ . If  $s'$  is another section of  $L|_W$ , then  $s' = fs$ , where  $f \in K(W)^*$ ; in order to have a well defined  $c_1(L)$ , we need to get rid of  $(f)_0 - (f)_\infty$ . This is exactly what we do to define cycle classes.

**1.2 DEFINITION.** We define  $R_d(X) \subseteq Z_d(X)$  as the subgroup generated by  $(f)_0 - (f)_\infty$  for every  $(d+1)$ -dimensional subvariety  $W \subseteq X$  and for every  $f \in K(W)^*$ .

**1.3 DEFINITION.** We define  $A_d(X)$  to be the  $d$ -th *Chow group* of  $X$ , as

$$A_d(X) := Z_d(X) / R_d(X).$$

**1.4 PROPOSITION.**

1. We can define  $c_1(L): A_{d+1}(X) \rightarrow A_d(X)$  as we unsuccessfully tried to do above with cycle groups;
2. proper pushforwards and flat pullbacks pass to the quotient, and commute with  $c_1(L)$ .
3. if  $\pi: E \rightarrow X$  is a rank  $r$  vector bundle, then there exists a well defined homomorphism  $\pi^*: A_d(X) \rightarrow A_{d+r}(E)$  that is an isomorphism.

**1.5 COROLLARY.** If  $s: X \rightarrow E$  is a section of a rank  $r$  vector bundle, then we can define  $s^!: A_d(E) \rightarrow A_{d-r}(X)$  to be the inverse of  $\pi^*$ .

**1.6 DEFINITION.** Let  $i: X \rightarrow P$  be a regular embedding of codimension  $r$  (i.e., locally, the ideal of  $X$  is generated by a regular sequence of length  $r$ ). Hence  $\mathcal{I}_X/\mathcal{I}_X^2$  is locally free of rank  $r$  and we can define the *Gysin pullback*  $i^!$  in this way: for every  $d$ -dimensional variety  $W \subseteq P$ , consider the cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & W \\ \downarrow & & \downarrow \\ X & \xrightarrow{i} & P \end{array}$$

note that we have to define  $i^![W] \in A_{d-r}(V)$ ; let

$$C_{V/W} := \text{Spec} \bigoplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1} \rightarrow V,$$

where  $J = \mathcal{I}_{V/W}$ . Now, we have a map  $q^*\mathcal{I}_X/\mathcal{I}_X^2 \rightarrow I/J^2$  and a diagram

$$\begin{array}{ccc} C_{V/W} & \longrightarrow & q^*N_{X/P} \\ & \searrow & \nearrow \\ & V & \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \quad \begin{array}{c} \\ \\ s_0 \\ \end{array}$$

where  $s_0$  is the zero-section. Finally, we define  $i^![W] := s_0^![C_{V/W}]$ .

We reason to use this definition is the following.

**1.7 DEFINITION.** A morphism  $f: X \rightarrow Y$  is *lci* of relative dimension  $r$  if locally there exists a factorization  $f = p \circ i$  with  $i: X \rightarrow P$  a proper embedding of codimension  $s$ , and  $p: P \rightarrow Y$  a smooth morphism of relative dimension  $r + s$ .

**1.8 THEOREM.** Let  $f: X \rightarrow Y$  be a global lci morphism of relative dimension  $r$ . Then  $f^! = i^! \circ p^*$  does not depend on the choice of  $p$  and  $i$ . Moreover,  $f^!$  is functorial and commuted with proper pushforwards, flat pullbacks, and Chern classes of vector bundles. In fact, for every cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

$f$  induces a homomorphism  $f^!: A_d(Y') \rightarrow A_{d+r}(X')$ . One says that  $f^! \in A^r(X \xrightarrow{f} Y)$  is a bivariant class.

**1.9 REMARK.** The Gysin pullback, with the same definition, works in the following context:  $i: X \rightarrow P$  a closed embedding and  $\mathcal{E}$  a rank  $r$  locally free sheaf on  $X$  with a surjection  $\mathcal{E} \rightarrow \mathcal{I}_X/\mathcal{I}_X^2$  (we replace  $N_{X/P}$  by  $E := \text{Spec Sym } \mathcal{E}$ , and we get  $i_E^! \in A^r(X \rightarrow P)$ ).

There are some directions we want to explore.

1. In Theorem 1.8, the global factorization seems a big hypothesis (even if it is verified in many situation. We can ask ourselves if we can define  $f^!$  without a *global* lci morphism.
2. We may want to extend Theorem 1.8 as much as possible to algebraic stacks.
3. We want to find a common generalization of Theorem 1.8 and Remark 1.9. The answer will be *virtual pullbacks*.

### 1.2 Algebraic stacks

Algebraic stacks are in some way a natural extension of schemes, as long as one consider the right definition of scheme. Indeed, defining algebraic stacks

starting from the definition of schemes as locally ringed topological spaces is kind of unnatural, but we can define schemes also in the following way.

We can associate to a scheme  $X$  its functor of points  $h_X: \mathcal{A}ff^{op} \rightarrow \mathcal{S}ets$  (where  $\mathcal{A}ff$  is the category of finite type affine schemes over  $K$ ). We define  $h_X(S) := \text{Mor}(S, X)$  and for every  $f: X \rightarrow Y$ ,  $h_X(f): h_X(S) \rightarrow h_Y(S)$  as the composition with  $f$ . This is the restriction of the complete functor of points that goes from the opposite category of all schemes; this latter functor is fully faithful by Yoneda, but one proves that also the one we considered before is fully faithful.

So, we can define schemes as a full subcategory of  $\text{Fun}(\mathcal{A}ff^{op}, \mathcal{S}ets)$ . The actual properties that the functors have to satisfy to enter in the subcategory of schemes differ when we change the kind of schemes we are interested in. But there are some properties that have to be respected for any definition of schemes we may want.

1. The sheaf property in the Zariski (étale, smooth, fppf) topology (also called descent property): for every diagram

$$\begin{array}{ccc} S_2 & \xrightarrow{p_2} & S_1 \\ p_1 \downarrow & & \downarrow \\ S_1 & \xrightarrow{\text{sm}} & S \end{array}$$

with  $g: S_1 \rightarrow X$  and  $p_1 \circ g = p_2 \circ g$ , there exists a unique  $f: S \rightarrow X$  inducing  $g$ .

2. There exists an open cover by affine schemes.

**1.10 REMARK.** In the definition of schemes, we can replace the second condition by the requirement of the existence of an étale affine open cover. With this changes, we are defining *algebraic spaces* as defined by Artin.

**1.11 DEFINITION.** An *algebraic stack* is a pseudofunctor  $\mathcal{A}ff^{op} \rightarrow \mathcal{G}roupoids$  such that:

1. the sheaf property holds (one says this as *being a stack*);
2. there exists an étale affine open cover (*Deligne-Mumford stack*) or a smooth affine open cover (*Artin stacks*).

A third definition with a flat affine open cover may be used, but Artin proved that this is equivalent to the smooth affine open cover.

Note that this definition is not a real definition because there are other properties that an algebraic stack has to satisfy; we just wrote those to highlight the similarities with the definition of schemes.

**1.12 EXAMPLE.** Consider the functor  $M_g^f: \mathcal{A}ff^{op} \rightarrow \mathcal{S}ets$  where  $M_g^f(S)$  is the set of morphisms  $\pi: C \rightarrow S$  that are smooth, projective, of relative dimension

1, with each fiber a connected smooth genus  $g$  curve, modulo isomorphisms, given by cartesian diagrams

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

There are two problems:

1. the first is that if  $S = U \cup V$  is an open cover for the functor, with associated morphisms  $C_U \rightarrow U$  and  $C'_V \rightarrow V$ , then to glue we need to choose an isomorphism  $C_{U \cap V} \cong C'_{U \cap V}$ ; but changing this isomorphism change the resulting glued object;
2. the second is that if we have an open cover  $S = U_1 \cup U_2 \cup U_3$ , then we can glue the corresponding families if and only if a cocycle condition is satisfied.

In order to solve these problems, we have to define  $M_g: \mathfrak{Aff}^{\text{op}} \rightarrow \mathfrak{Groupoids}$ , where  $M_g(S)$  is defined in the same way as before, but without modding by the isomorphisms, that are remembered by the groupoid structure. It turns out that the so defined  $M_g$  is an algebraic stack, smooth, connected, of dimension  $3g - 3$ , Deligne-Mumford if and only if  $g \geq 2$ .

1.13 REMARK. If  $X$  is an Artin stacks, and  $x$  a geometric point (i.e., an object of the groupoid  $X(L)$  for an algebraically closed field  $L$ ), then by the property of having a smooth affine open cover, we have that  $\text{Aut}(x)$  is an algebraic group. If  $X$  is DM, then  $\text{Aut}(x)$  is étale (that in characteristic 0 implies that  $\text{Aut}(x)$  is finite). For any  $X$ , the condition of  $\text{Aut}(x)$  being étale defines an open substack (basically, the maximal DM substack of  $X$ ).

1.14 EXAMPLE. Consider  $\mathfrak{M}_g$ , the functor defined as before, but allowing the fibers to be nodal (not necessarily stable) and changing “genus” to “arithmetic genus”. This is an algebraic stack, contains  $M_g$  as an open substack and it is smooth, connected, of dimension  $3g - 3$ . If we let  $\overline{M}_g$  to be the DM locus in  $\mathfrak{M}_g$ , then  $\overline{M}_g$  is empty if  $g = 0$ , and a proper DM stack if  $g \geq 1$ .

1.15 EXAMPLE. We can consider  $\mathfrak{M}_{g,n}$  the functor defines as  $\mathfrak{M}_g$ , but where each family  $C \rightarrow S$  have sections  $s_1, \dots, s_n$  such that for every fiber  $C_p$ , the points  $s_i(p)$  are distinct and lie in the smooth part of  $C_p$ . Morphisms are required also to commute with the data of the sections. It turns out that  $\mathfrak{M}_{g,n}$  is smooth, connected, locally of finite type, of dimension  $3g - 3 + n$ ; inside it, the open locus where it is DM (in this locus the automorphisms groups are finite, even in positive characteristic, thanks to a geometric argument) is denoted  $\overline{M}_{g,n}$  and it is proper and non empty if and only if  $2g - 2 + n > 0$ .

Lecture 2 (1 hour)  
October 5<sup>th</sup>, 2010

1.16 REMARK. For morphisms of algebraic stacks, there is a valuative criterion for properness and separatedness similar to the one that works for scheme.

For example,  $f: X \rightarrow Y$  is proper if and only if for every DVR  $R$ , there is a finite cover  $\text{Spec } \tilde{R} \rightarrow \text{Spec } R$  and a dashed morphism

$$\begin{array}{ccccc} \tilde{\eta} & \longrightarrow & \eta & \longrightarrow & X \\ \downarrow & \dashrightarrow & \downarrow & \dashrightarrow & \downarrow f \\ \text{Spec } \tilde{R} & \longrightarrow & \text{Spec } R & \longrightarrow & Y \end{array}$$

that commutes with the diagram.

1.17 REMARK. Groupoids are a 2-category, therefore also algebraic stacks form a 2-category. Hence when we say that a diagram commutes we really mean that they 2-commute, and fiber products are 2-fiber products.

### 1.3 Key example

Let  $X$  be a scheme and  $G$  an algebraic group acting on  $X$ . Define  $[X/G]$  to be the stack quotient by letting  $[X/G](S)$  be the set of diagrams

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$$

where  $P \rightarrow S$  is a  $G$ -torsor and  $P \rightarrow X$  is  $G$ -equivariant, and the morphisms are isomorphisms of  $G$ -torsors, commutative with the morphisms to  $X$ . One can define pullbacks in a natural way.

This mimics what we do when we define the quotient groupoid of the action of a group  $G$  on a set  $X$ , where the objects are the elements of  $X$ , and the morphisms between  $x$  and  $y$  are the elements of  $G$  sending  $x$  to  $y$ . In order to glue this local constructions, we have to use the torsors as we did globally.

Moreover, there is a 2-cartesian diagram

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & [X/G] \end{array}$$

and the stack quotient behaves as if  $G$  were acting freely.

1.18 EXAMPLE. Consider  $a_0, \dots, a_n \in \mathbb{Z}_{>0}$ , and the stack quotient

$$[\mathbb{A}^{n+1} \setminus \{0\} / \mathbb{G}_m],$$

over  $K = \bar{K}$ , where  $\lambda \cdot (x_0, \dots, x_n) := (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$ . The quotient is the weighted projective space  $\mathbb{P}(a_0, \dots, a_n)$ . It is a proper DM stack if and only if each  $a_i$  is prime to the characteristic of  $K$ . One can prove that, over the complex numbers,  $\mathbb{P}(4, 6) \cong \bar{M}_{1,1}$ .

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## 2 MIXING

### 2.1 Intersection theory for algebraic stacks

Starting in the end of the eighties, intersection theory has been extended to a broader setting, DM stacks in the beginning and during the years also to Artin stacks. The following theorem sums up the results that allow us to use intersection theory with algebraic stacks almost like we do with schemes.

**2.1 THEOREM** (Vistoli, Edidin-Graham (Totaro), Kresch). *The Chow groups  $A_*$  can be defined for a large class of algebraic stacks. Flat pullbacks work, proper pushforwards work with rational coefficients (because the degree of a 0-cycle on a stack is in general a rational number). The Gysin map for regular embeddings (hence for global lci morphisms) also works.*

### 2.2 Algebraic stacks for intersection theory

Let  $X, Y$  be schemes,  $f: X \rightarrow Y$  a morphism that factors as a regular embedding  $i: X \rightarrow P$  and a smooth morphism  $p: P \rightarrow Y$ . For  $W \rightarrow Y$ , consider the cartesian diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & P & \xrightarrow{p} & Y \\ q \uparrow & & \uparrow & & \uparrow \\ V & \longrightarrow & W_p & \longrightarrow & W \end{array}$$

Then we have  $f^! [W] := s_0^! [C_{V/W_p}]$  for  $s_0: V \rightarrow q^* N_{X/P}$ . But if  $f$  is already a regular embedding, then  $f^! [W] = \tilde{s}_0^! [C_{V/W}]$  with  $\tilde{s}_0: V \rightarrow q^* N_{X/Y}$ . In this case, Fulton proved that there are natural exact sequences of cones, for examples

$$0 \rightarrow i^* T_{P/Y} \rightarrow N_{X/P} \rightarrow N_{X/Y} \rightarrow 0,$$

that stays exact when we pullback to  $V$  via  $q$ . A second exact sequence is

$$0 \rightarrow q^* i^* T_{P/Y} \rightarrow C_{V/W_p} \rightarrow C_{V/W} \rightarrow 0,$$

and we have injective morphisms from the latter to the former.

When we remove the assumption that  $f$  is a regular embedding, we still have  $0 \rightarrow i^* T_{P/Y} \rightarrow N_{X/P}$ , but this now continues to  $I/I^2 \rightarrow \Omega_{P/Y}|_X$ . Note that

$$\underbrace{[I/I^2]_{-1}} \rightarrow \underbrace{[\Omega_{P/Y}|_X]_0} \cong \tau_{\geq 1} L_f^\bullet$$

is the truncation of the cotangent complex.

Let us recall some properties of the cotangent complex. When we have a composition  $h: X \xrightarrow{f} Y \xrightarrow{g} Z$ , then we have a sequence

$$f^* \Omega_g \rightarrow \Omega_h \rightarrow \Omega_f \rightarrow 0;$$

we interpret the lack of a zero on the left as the effect of the distinguish triangle

$$f^*L_g^\bullet \rightarrow L_h^\bullet \rightarrow L_f \xrightarrow{+1}$$

in  $D_{\text{coh}}^{\leq 0}$  inducing the sequence, where  $h^0(L_h^\bullet) = \Omega_h$  and  $h^0(L_f^\bullet) = \Omega_f$ .

Therefore,  $f: X \rightarrow Y$  is lci if and only if  $L_f^\bullet$  is perfect (of perfect amplitude contained in)  $[-1, 0]$ , and locally  $L_f^\bullet$  is isomorphic to  $\mathcal{E}^{-1} \rightarrow \mathcal{E}^0$  with  $\mathcal{E}^i$  locally free of finite rank.

**2.2 DEFINITION.** The normal vector bundle stack is  $\underline{N}_{X/Y} = \underline{N}_f := [E_1/E_0]$ , where  $E_i = \text{Spec Sym } \mathcal{E}^i$  (note that  $E_0 \rightarrow X$  is a sheaf of abelian groups).

This works in a much larger generality.

**2.3 THEOREM.** *There exists a notion of abelian cone stack (i.e., that is locally the stack quotient of  $\text{Spec Sym } \mathcal{F}$  by a vector bundle, where  $\mathcal{F}$  is a coherent sheaf), and there exists an equivalence of categories*

$$D_{\text{coh}}^{[-1,0]}(X) \rightarrow \text{Ho}(\text{abelian cone stacks over } X),$$

mapping complexes perfect in  $[-1, 0]$  to vector bundle stack.

**2.4 REMARK.** Deligne in exposé 18, SGA4, has a similar theorem which gives an equivalence between the derived category of abelian sheaves over  $X$  to the homotopy category of Picard stacks over  $X$ . To define Picard stacks, Deligne does not start with the usual definition of groups replacing sets by groupoids, but with a slightly different definition, that allows himself to keep track of as less 2-arrows as possible. Indeed, he insists in associativity and commutativity, but after that he requires that for every  $g \in G$ ,  $\cdot g: G \rightarrow G$  is an isomorphism. In the groupoids setting, this translates naturally to the requirement that  $\cdot g$  are equivalences.

**2.5 REMARK.** It is easy to see that there exists an equivalence of categories from the opposite of coherent sheaves over  $X$  to abelian cones over  $X$ , sending  $\mathcal{F}$  to  $\text{Spec Sym } \mathcal{F}$ .

**2.6 COROLLARY.**

1. For every lci morphism  $f: X \rightarrow Y$  of relative dimension  $r$ , the abelian cone stack associated to  $L_f^\bullet$  is a vector bundle stack of rank  $r$ .
2. for every cartesian diagram

$$\begin{array}{ccc} V & \longrightarrow & W \\ q \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

the abelian cone stack  $\underline{N}_{V/W}$  associated to  $\tau_{\geq -1}L_{V/W}^\bullet$  injects into  $q^*\underline{N}_{X/Y}$ ; more-



over there is a natural closed embedding

$$\underline{C}_{V/W} \hookrightarrow \underline{N}_{V/W} \hookrightarrow q^* \underline{N}_{X/Y}.$$

**2.7 THEOREM.** *Let  $f: X \rightarrow Y$  be a DM type morphism of algebraic stacks which is lci, and let  $W$  be a variety of dimension  $d$  (or more generally a pure dimensional algebraic stack of dimension  $d$ ) with the cartesian diagram*

$$\begin{array}{ccc} V & \longrightarrow & W \\ q \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

then  $f^! [W] \in A_{d+r}(V)$  is defined to be  $s_0^!([\underline{C}_{V/W}])$  with  $s_0: V \rightarrow q^* \underline{N}_{X/Y}$  the zero section.

Lecture 3 (1 hour)  
October 7<sup>th</sup>, 2010

We can sum up what we did (in the contest of scheme) in the following way.

**2.8 THEOREM.** *For every closed embedding  $i: X \rightarrow Y$ , let  $\underline{N}_{X/Y} = \text{Spec Sym } I/I^2$  and  $\underline{C}_{X/Y} = \text{Spec } \bigoplus I^n/I^{n+1}$  (recall that there is a closed embedding  $\underline{C}_{X/Y} \hookrightarrow \underline{N}_{X/Y}$ ), then:*

1. if  $Y$  is of pure dimension  $d$ , then so is  $\underline{C}_{X/Y}$ ;
2. for every cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{j} & W \\ q \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

we have  $\underline{C}_{V/W} \hookrightarrow q^* \underline{C}_{X/Y} \hookrightarrow q^* \underline{N}_{X/Y}$ ;

3. for every  $\underline{N}_{X/Y} \hookrightarrow E$ , with  $E$  a rank  $r$  vector bundle (equivalently,  $\mathcal{E} \rightarrow \mathcal{I}/\mathcal{I}^2$ ), we get  $i_E^! \in A^r(X \rightarrow Y)$ .

Previously, we have seen the translation to algebraic stacks. If  $f: X \rightarrow Y$  is a DM type morphism of algebraic stacks, then we have  $\underline{N}_{X/Y}$ , the abelian cone stack associated to  $\tau_{\geq -1} L_f^\bullet$ , that contains as a close substack  $\underline{C}_{X/Y}$ , as induced by Fulton's construction. Moreover, the first two properties hold unchanged, while in the third we require  $\underline{N}_{X/Y} \subseteq \underline{E}$ , with  $\underline{E}$  a vector bundle stack, or equivalently  $\varphi \rightarrow \tau_{\geq -1} L_f^\bullet$  perfect of rank  $r$ , and we get  $f_E^! \in A^r(X \rightarrow Y)$ .

More analogies hold; for example there is a degeneration to the normal cone.

**2.9 PROPOSITION (Manolache).** *The map  $f_E^!$  is functorial.*

**2.10 COROLLARY.** *We get an easy proof of Costello's pushforward (that has enumerative geometry applications).*

2.11 DEFINITION.

1. The morphism  $f_{\underline{E}}^1$  is called the *virtual pullback* defined by the obstruction theory  $(\varphi, \mathcal{E}^\bullet)$ .
2. If  $Y$  is pure dimensional of dimension  $d$ , then  $[X]^{\text{vir}} := f_{\underline{E}}^1[Y] \in A_{d+1}(X)$  is called the *virtual fundamental class*.

2.3 How does one apply this?

If  $M$  and  $\mathfrak{M}$  are moduli spaces with a forgetful map  $f: M \rightarrow \mathfrak{M}$ , and  $p \in M$ , then from deformation theory we have three vector spaces  $T_p^i$  with  $i \in \{0, 1, 2\}$ .

For example, if  $\mathfrak{M}$  is just one point and  $M$  is the moduli space of smooth projective varieties, and  $p \in M(K = \bar{K})$  corresponds to a variety  $V$ , then we have  $T_p^1 = H^1(V, T_V)$ . The idea now is that  $f$  is of DM type at  $p$  if and only if  $T_p^0 = 0$ . In particular, if  $T_p^0 = 0$  for every  $p \in M$ , then the expected dimension of  $M$  is  $\dim T_p^1 - \dim T_p^2$ , which is constant and we get an obstruction theory.

The aim is to use virtual classes to define numerical invariants. To start, one need a proper moduli space.

2.12 EXAMPLE. Let  $M_{c_1, c_2}^2$  be the moduli space of surfaces of general type over  $\mathbb{C}$  with given invariants; for every surface  $V$ ,  $T_{[V]}^0 = H^0(V, T_V) = 0$ , hence we can compute  $\dim T^1 - \dim T^2$  using Riemann-Roch. The problem is that  $M_{c_1, c_2}^2$  is not proper, and there are no clues on how to extend the obstruction theory to the border.

The situation is better in the moduli space of maps over the complex number. Recall that we have that  $\bar{M}_{g,n} \subseteq \mathfrak{M}_{g,n}$  is the inclusion of the DM locus. One can prove that  $\bar{M}_{g,n}$  is proper, in any characteristic using stable reduction, or over  $\mathbb{C}$  using the minimal models of surfaces of Castelnuovo.

Consider a smooth projective variety  $V$ , and fix a continuous homomorphism  $d: \text{Pic}(V) \rightarrow \mathbb{Z}$ . We can then define  $\mathfrak{M}_{g,n}(V, d)$ : the objects over  $S$  are tuples  $(C, \pi, s_i, f)$ , with  $(C, \pi, s_i) \in \mathfrak{M}_{g,n}(S)$ , and  $f: C \rightarrow V$ , such that  $d = \deg_{\mathbb{C}} f^*$  fiberwise; the morphisms are morphisms on the curves that commute with every data.

Then, we define  $\bar{M}_{g,n}(V, d)$  as the DM locus of this stack, and one can prove with the same methods that this is proper. Moreover, one get that at the point  $(C, \pi, s_i, f)$ , we have  $T^0 = 0$  and  $T^i = H^{i-1}(C, f^*T_V)$ .

From the work of Chen and Ruan in the symplectic case, Abramovich, Vistoli, and others in the algebraic case, we have a generalization of Gromov-Witten invariants, where we replace  $V$  with a smooth projective DM stack. The obstruction theory is the same, and to get properness, we have to replace  $\mathfrak{M}_{g,n}$  with  $\mathfrak{M}_{g,n}^{\text{tw}}$ , where we allow stacky points along  $s_i$  and at nodes.

In the case that  $V$  is not projective, in some cases one can do the same when there is a torus action on  $V$ . In this situation, the positive dimensional orbits are all tori, so they give no contribution to integrals having Euler characteristic 0. To make this more precise, one use localization theorem, that states that one

can work only with the fix locus of the action. The virtual localization theorem state that the same works with virtual pullbacks and virtual fundamental classes.

If  $V$  is singular, Jun Li extended to algebraic geometry the degeneration formula (first found in symplectic geometry). The idea is to degenerate one complicated variety to two simpler varieties meeting transversally along a divisor. More precisely, one construct a proper flat morphism  $W \rightarrow B \ni b_0$ , smooth over  $B \setminus \{b_0\}$ , with  $W$  smooth and  $B$  a smooth curve; all such that the generic fiber is  $V$  and the special fiber is the reducible one.

With this formula, one can define  $\overline{M}_{g,n}(W/B, d)$ , where objects are morphisms  $C \rightarrow S$  commuting with  $W \rightarrow B$ . In particular,  $\overline{M}_{g,n}(W_b, d)$  is composed of the fibers over  $b$  as before, if  $b \neq b_0$ ; instead, if  $b = b_0$ ,  $\overline{M}_{g,n}(W_b, d)$  is proper, and  $H^1(C, f^*T_{W_b})$  is replaced by  $\text{Ext}^1(f^*L_{W_b}^\bullet, \mathcal{O}_C)$ . The problem is that there may be a  $T^3$  coming from the  $\text{Ext}^2$ , if a component of  $C$  maps to  $D$ . The idea of Jun Li is to change compactification: when problems arise, we can blow up such component; in practice, this can be achieved blowing up  $D$  inside  $W$ . The trick is to allow blown up maps, but not too much; this different compactification yields a smooth Artin stack.

A problem is that to get properness, he has to impose *predeformability* conditions. Let  $C = C_1 \cup C_2$  be a reducible curve; then he impose that the only thing that can be mapped to  $D$  are nodes (i.e., each component of the curve goes either to  $V_1$  or to  $V_2$ ). Moreover,  $\text{mult}_p f^{-1}(D)$  has to be the same in  $C_1$  and in  $C_2$ . This condition implies that, at least locally, the map deforms to the nearby smooth fibers. This approach had been pursued also by Caporaso and Harris in the definition of Gromov-Witten invariants for  $\mathbb{P}^2$  and its blown ups.

We have seen that obstruction theory comes from deformation theory, in particular from the deformation theory of the map  $\mathfrak{M}_{g,n}(V, d) \rightarrow \mathfrak{M}_{g,n}$ . When we consider  $\overline{\mathfrak{M}}_{g,n}(V, d) \subseteq \mathfrak{M}_{g,n}(V, d)$ , this is open and to does not pose problems. Instead, predeformability is not an open condition. This is the main technical problem of the paper, and Jun Li solve this using log geometry.

Another approach is to use moduli spaces of sheaves, where over reasonable assumptions one has  $T_{[\mathcal{F}]}^i = \text{Ext}^i(\mathcal{F}, \mathcal{F})$ . A technical problem is that  $T^0$  is always non zero, but this can be solved using rigidification, or taking determinants. In this way, considering simple sheaves over a smooth proper surface, one constructs algebraic Donaldson invariants. In the threefold case, we have Donaldson-Thomas invariants, and also Pandharipande-Thomas. The relations amongst these invariants is largely unknown.