Deformations of fibrations
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Workshop in deformation theory II, Roma La Sapienza

August 30th, 2010 – September 3rd, 2009

Contents

1 Introduction

Lecture 1 (1 hour)
August 30th, 2010

We will apply deformation theory to study and discover some nontrivial birational properties of moduli spaces. In particular we will consider the moduli space of curves. So in this first lecture we will not talk about deformation theory, but describe the moduli space setup.

We will work over $\mathbb{C}$; for any $g \geq 3$, we consider $\overline{M}_g$, the coarse moduli space of stable curves of genus g; it is a projective compactification of the coarse moduli space M_g of smooth curves of genus g. We are not really interested on what "stable" means; the only thing we need is that the compactification is modular.

The space $\overline{M}_g$ is normal, irreducible, of dimension $3g - 3$. It has a "weak" universal property: for every family $\mathcal{C} \to S$ (i.e., any flat morphism whose fibers are stable curves of genus g), there is an induced morphism $\psi_S \colon S \to \overline{M}_g$ such that $\psi_S(s) = [\mathcal{C}(s)]$. We called this "weak" because we don't have in general the converse: given $\psi \colon S \to \overline{M}_g$, in general there does not exist a family $\mathcal{C} \to S$ such that $\psi = \psi_S$. This problem arise when the curves in the family have nontrivial automorphisms group.

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1.1 fact. There is a nonempty open subscheme $\overline{M}_g^\circ \subseteq \overline{M}_g$ and a (universal) family $\mathcal{C}^\circ \to \overline{M}_g^\circ$ such that for every morphism $\psi\colon S \to \overline{M}_g^\circ$, there exists a family $f\colon \mathcal{X} \to S$ with $\psi_f = \psi$.

Note that we introduced the coarse moduli space of curves, but we are really working on the moduli space (stack) of curves, since we work with families all the time.

1.1 *Generic curves*

1.2 definition. Let $f\colon \mathcal{C} \to S$ be a family of curves of genus g with S integral. Let $\psi_S\colon S \to \overline{M}_g$ the induced morphism. We say that f has *general moduli* if ψ_f is dominant.

1.3 definition. A *general curve* of genus g (or a *curve with general moduli*) is a general point of $\overline{M}_g$.

So, if we have a generic point in the base of a family S with general moduli, we also have a generic curve. We still have the problem to choose a generic point of S; but with curves is easier, as in the following example.

1.4 example. Let $g = 3$ and $C \subseteq \mathbb{P}^2$ a nonsingular quartic. This gives us a generic curve of genus g. But if we consider the quartic given by $x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0$, that is the Klein's quartic (a very specific quartic with 168 automorphisms), this is not a generic quartic. Anyway, when we see the Klein's quartic inside the previous family, it is still a generic curve.

1.5 proposition. *Let C be a general nonsingular curve of genus g. Then the following hold.*

1. *for every $L \in \mathrm{Pic}(C)$, the natural map $\mu_L\colon\ \mathrm{H}^0(L) \otimes \mathrm{H}^0(\omega_C L^{-1}) \to \mathrm{H}^0(\omega_C)$ is injective (this is called Petri's conjecture, proved by Gieseker). In particular, for every L of degree d with $\mathrm{h}^0(L) = r+1$, the Brill-Noether number $\varrho := g-(r+1)(g-d+2)$ is non-negative.*
2. $\mathrm{h}^1(L^2) = 0$ *for every L with* $\mathrm{h}^0(L) \geq 2$ *(this is a consequence of the previous statement).*
3. *C does not have irrational involutions (i.e., a nontrivial morphisms to an irrational curve).*
4. *C does not have nontrivial automorphisms.*

Note that this proposition is actually a collection of important theorems. We state two special cases of the inequality of the Brill-Noether number. If $r = 1$, we have $d \geq 1/2g + 1$; if $r = 2$, then $d \geq 2/3g + 2$.

1.2 *Birational geometry of the moduli spaces of curves*

Let us review now some notions in the birational geometry of the coarse moduli space of curves: $\overline{M}_g$ is

1. rational for $0 \leq g \leq 6$;
2. unirational for $7 \leq g \leq 14$;
3. rationally connected for $g = 15$;
4. of Kodaira dimension $-\infty$ for $g = 16$;
5. of Kodaira dimension at least 2 for $g = 23$;
6. of general type for $g = 22$ or $g \geq 24$.

In particular, note that we don't know anything for $17 \leq g \leq 21$. This is the motivation behind these lectures.

Unirationality is a weaker property than rationality, but from our viewpoint it has the advantage that can be studied with families: a moduli space is unirational if and only if there is a family with generic moduli with a rational base.

1.6 definition. An integral variety M is *uniruled* if there is an integral Y with $\dim Y = \dim M - 1$ and a dominant morphism $Y \times \mathbb{P}^1 \to M$.

So a variety is uniruled if for any generic point passes a line. Our aim is to study this notion of uniruledness for $M = \overline{M}_g$. Similarly to rationality versus unirationality, uniruledness is a weaker property than ruledness that can be studied with families.

1.3 *Fibrations*

1.7 definition. A *fibration* is a morphism $f\colon X \to S$ from a projective nonsingular surface X to a projective connected nonsingular curve S which is surjective and has connected fibers.

As a notation, we use g for the genus of the general (smooth) fiber of f, and b for the genus of the base S.

1.8 definition. Let f be a fibration. We say that f is:

- *relatively minimal* if X contains no (-1)-curves in any fiber;
- *semistable* if all fibers have at most ordinary double points (i.e., nodes);
- *isotrivial* if any two general fibers are isomorphic.

Recall that $E \subseteq X$ is a (-1)-curve if it is irreducible nonsingular and with $E^2 = -1$.

Let $f\colon X \to S$ be a fibration. A priori the induced morphism ψ_f is defined only over a nonempty open set of S; but since S is a curve and $\overline{M}_g$ is proper, ψ_f can be extended uniquely to a morphism $S \to \overline{M}_g$ that we will call also ψ_f. This morphism is non-costant if and only if f is non-isotrivial.

1.9 definition. A *rational fibration* is a fibration of the form $f\colon X \to \mathbb{P}^1$.

1.10 example. Let Y be a nonsingular surface and $C \subseteq Y$ a nonsingular curves. Assume that $\dim |C| \geq 1$ (i.e., the curve moves in its linear system). Let $\Lambda \subseteq |C|$ be a pencil containing C. This pencil defines a rational map $Y \dashrightarrow \mathbb{P}^1$; if we blow up enough times Y, we have a morphism $X \to \mathbb{P}^1$ from a variety which is birational to Y.

1.11 proposition. *Assume that C is a connected nonsingular curve of genus g, moving in a nontrivial linear system $|C|$ in a surface Y. Let $\Lambda \subseteq |C|$ be a pencil containing C. Assume that:*

1. *C has no nontrivial automorphisms;*
2. *C does not have rational involutions.*

Then the following are equivalent:

1. *Λ defines an isotrivial fibration;*
2. *Y is birationally equivalent to $C \times \mathbb{P}^1$;*
3. *Y is a non-rational ruled surface.*

Proof.

$1 \Rightarrow 2)$ Let $X \to \mathbb{P}^1$ the morphism constructed before, where X is birationally equivalent to Y. By the structure theorem for isotrivial fibrations, there is a nonsingular curve Γ and a finite group G acting on Γ and on C such that X is birational to ${}^{C \times \Gamma}/_{G}$, and the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & {}^{C \times \Gamma}/_{G} \arrow[d] \\ \mathbb{P}^1 \arrow[r, equal] & {}^{\Gamma}/_{G}. \end{tikzcd}$$

But in our case, the action of G on C is trivial, because there are no nontrivial automorphisms of C by assumptions; therefore X is birational to $C \times {}^{\Gamma}/_{G} \cong C \times \mathbb{P}^1$.

$2 \Rightarrow 3)$ Obvious.

$3 \Rightarrow 1)$ Let D be a generic member of Λ. Then we have the diagram

$$\begin{tikzcd} D \arrow[rr, "\Phi"] \arrow[dr, bend right] & & \Gamma \\ & Y \dashrightarrow \Gamma \times \mathbb{P}^1, \arrow[ur, bend right] & \end{tikzcd}$$

so Φ is a noncostant irrational involution; by the second assumption, Φ is an isomorphism. Hence $\Gamma \cong C$ and the fibration is isotrivial. ☐

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1.12 corollary. *The following are equivalent:*

1. *$\overline{M}_g$ is uniruled;*

2. *a general C of genus g moves in a nontrivial linear system on some non-ruled surface.*

Proof.

1 $\Rightarrow$ 2) We will assume that every morphism to the moduli space corresponds to a family. This is, as we said, not true in general for the coarse moduli space, but it is for the moduli stack. If $\overline{M}_g$ is uniruled, then there exists a dominant $\psi_f \colon Z \times \mathbb{P}^1 \to \overline{M}_g$; with our assumption, such a morphism induces a family $f \colon \mathcal{X} \to Z \times \mathbb{P}^1$. Let $z \in Z$ be a general point; the restriction of f to $\{z\} \times \mathbb{P}^1$ is a morphism $\mathcal{X}_z \to \mathbb{P}^1$, that gives us the pencil we needed.

2 $\Rightarrow$ 1) Consider a general curve $[C] \in \overline{M}_g$ such that $C \subseteq Y$ and $\dim |C| \geq 1$. Let $\Lambda \subseteq |C|$ be a pencil and $f \colon \mathcal{X} \to \mathbb{P}^1$ the associated rational fibration. Since C is general it has no nontrivial automorphisms or rational involutions; then, by proposition 1.11, the family is non-isotrivial. Using the assumption that every morphism to $\overline{M}_g$ corresponds to a family, this would already be the proof, because f would induce a morphism $\mathbb{P}^1 \to \overline{M}_g$ passing through $[C]$. But we could do better: note that to give a general curve C in $\overline{M}_g$ moving in a nontrivial linear system on some non-ruled surface is equivalent to have a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "f"'] & \mathcal{Y} \arrow[dl, "\beta"] \\ V & \end{tikzcd}$$

such that $\mathcal{Y} \to V$ is a family of surfaces, and for every $v \in V$, $|\mathcal{X}(v)|$ is a curve in $\mathcal{Y}(v)$. Let $\mathcal{L} = \mathcal{O}_{\mathcal{Y}}(\mathcal{X})$ and $\mathcal{E} = \beta_\star \mathcal{L}$, and assume for simplicity that $\operatorname{rk} \mathcal{E} = 2$. Then $\mathcal{E} = \mathcal{O}_V^2$ and $\mathbb{P}(\mathcal{E}) = V \times \mathbb{P}^1$. So we have a rational map $\mathcal{Y} \dashrightarrow V \times \mathbb{P}^1$ and hence, blowing up, a morphism $\widetilde{\mathcal{Y}} \to V \times \mathbb{P}^1$. This morphism is a family of surfaces, each of them mapping to $\mathbb{P}^1$. For every $v \in V$, the situation is the following:

$$\begin{tikzcd} Y := \mathcal{Y}(v) \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{Y}} \arrow[d] & \\ \{v\} \times \mathbb{P}^1 \arrow[r] & V \times \mathbb{P}^1 \arrow[r, "\text{dominant}"] & \overline{M}_g \end{tikzcd}$$

and the left vertical morphism is a rational fibration. $\square$

Recall that the Kodaira dimension of $\overline{M}_g$ is at least 2 for $g \geq 22$. Also, a uniruled variety has negative Kodaira dimension. Hence $\overline{M}_g$ cannot be uniruled for $g \geq 22$, and we have the following, that can be viewed as a theorem belonging to surface theory.

1.13 corollary. *A general curve of genus $g \geq 22$ cannot move in a positive dimensional linear system on any non-ruled surface.*

2 Numerical computations and deformation theory

2.1 *The sheaf* $\mathcal{E}\mathrm{xt}^1_f$

Suppose $f\colon X \to S$ is a fibration (not necessarily rational). We can associate to f the functor $\mathrm{Def}_f\colon \mathfrak{Art} \to \mathfrak{Sets}$ of infinitesimal deformations of f leaving the base fixed. An element of $\mathrm{Def}_f(A)$ is a diagram

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & \mathcal{X} \arrow[d] \arrow[dd, bend left=60, "\text{flat}"] \\ S \arrow[r] & S \times \operatorname{Spec} A \arrow[d] \\ & \operatorname{Spec} A \end{tikzcd}$$

(note that S is deformed trivially).

Since f defines a morphism $\psi_f\colon S \to \overline{M}_g$, we can think we are deforming ψ_f instead of f. Anyway, we will not pursue this viewpoint.

Consider the sequence

$$0 \to f^\star\omega_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0;$$

the first two terms are locally free, but the third is not in general, since the map f is not necessarily smooth (note that instead the relative dualizing sheaf $\omega_{X/S} = \omega_X \otimes f^\star\omega_S^{-1}$ is invertible).

Dualizing the sequence, we get

$$0 \to \mathrm{T}_X \to f^\star\mathrm{T}_S \to N \to 0,$$

where $N := \mathcal{E}\mathrm{xt}^1(\Omega^1_{X/S}, \mathcal{O}_X)$. If f has reduced fibers, N has finite support; in general N is supported on the singular locus of f.

2.1 theorem (Arakelov-Serrano). *The morphism f is non-isotrivial if and only if $f_\star\mathrm{T}_X = 0$. Moreover, $\mathrm{h}^1(X, \mathrm{T}_{X/S}) = 0$ if f is non-isotrivial and relatively minimal.*

Let $\mathcal{E}\mathrm{xt}^1_f$ be the first derived functor of $f_\star\,\mathcal{H}\mathrm{om}$ and consider $\mathcal{E}\mathrm{xt}^1_f(\Omega^1_{X/S}, \mathcal{O}_X)$. This is a sheaf over S such that, for every point $p \in S$,

$$\mathcal{E}\mathrm{xt}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) \otimes_p \mathbb{C} \cong \mathrm{Ext}^1(\Omega^1_{X(p)}, \mathcal{O}_{X(p)}),$$

and if $X(p)$ is nonsingular, then this is equal to $\mathrm{H}^1(X(p), \mathrm{T}_{X(p)})$, a vector space of dimension $3g-3$.

2.2 lemma. *There is an exact sequence of sheaves over S:*

$$0 \to \mathrm{R}^1 f_\star\mathrm{T}_{X/S} \to \mathcal{E}\mathrm{xt}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) \to f_\star N \to 0.$$

Moreover, the middle term is locally free of rank $3g-3$. Infact, $\mathcal{E}\mathrm{xt}^1_f(\Omega^1_{X/S}, \mathcal{O}_X) \cong$

$\mathcal{H}om(f_\star(\Omega^1_{X/S} \otimes \omega_{X/S}), \mathcal{O}_S)$ which is locally free being a dual.

2.3 PROPOSITION. *The spaces $H^i(\mathcal{E}xt^1_f(\Omega^1_{X/S}, \mathcal{O}_X))$ for $i = 0, 1$ are respectively tangent and obstruction spaces for* Def_f.

Sketch of the proof. We identify Def_f with $\mathrm{Def}\,\psi_f$. It is well known that tangent and obstruction spaces to the deformations of the morphism ψ_f, which is a morphism from a projective variety to a quasi-projective variety, are $H^i(\psi_f^\star T_{\overline{M}_g})$ for $i = 0, 1$.

Since we are currently thinking of $\overline{M}_g$ as the moduli stack, it is smooth of dimension $3g - 3$ and the tangent space is exactly the space we need. Otherwise, we can also prove directly this fact using cocycles. □

If f is non-isotrivial, by Theorem 2.1 $f_\star T_X = 0$, hence also $f_\star T_{X/S} \subseteq f_\star T_X$ is zero. Therefore, $H^1(f_\star T_{X/S}) = 0$.

We can interpret the sequence of Lemma 2.2 in this way: if f were a ramified cover of a curve, we would have Riemann existence theorem that would say that deforming the cover means deforming the branch points, and that we can deform the branch points arbitrarily. Instead, in our situation we cannot deform arbitrarily the singular points, because in general $H^1(R^1 f_\star T_{X/S})$ is not zero.

Lecture 3 (1 hour) September 2nd, 2010

Consider a non-isotrivial fibration $f\colon X \to S$ and the sheaf $\mathcal{E}xt^1_f(\Omega_{X/S}, \mathcal{O}_X)$; then the sequence

$$(1) \quad 0 \to f^\star\omega_S \to \Omega^1_X \to \Omega^1_{X/S} \to 0$$

induces the exact sequences

$$0 \to T_{X/S} \to T_X \to N = \mathcal{E}xt^1(\Omega_{X/S}, \mathcal{O}_X) \to 0, \text{ and}$$
$$0 \to R^1 f^\star T_{X/S} \to \mathcal{E}xt^1_f(\Omega_{X/S}, \mathcal{O}_X) \to f_\star N \to 0.$$

Our goal now is to compute the Euler characteristic of the sheaf $\mathcal{E}xt^1_f(\Omega_{X/S}, \mathcal{O}_X)$ using these sequences.

2.4 PROPOSITION. *We have*

$$\chi(\mathcal{E}xt^1_f(\Omega_{X/S}, \mathcal{O}_X)) = 11\chi(\mathcal{O}_X) - 2K_X^2 + 2(b-1)(g-1).$$

Proof. We have $\mathcal{E}xt^2_f(\Omega_{X/S}, \mathcal{O}_X) = 0$, because the fibers are curves and this is $\mathcal{E}xt^2$ on the fibers. So,

$$\begin{aligned}\chi(f_\star N) &= h^0(f_\star N) - h^1(f_\star N) = \\ &= h^0(N) - h^1(N) - \cancel{h^0(R^1 f_\star N)}\end{aligned}$$

where the last term is 0 because it is a part of the filtration of $\mathcal{E}xt^2_f(\Omega_{X/S}, \mathcal{O}_X)$ that vanish. So $\chi(f_\star N) = h^0(N) - h^1(N) = \chi(N)$.

Now,

$$\begin{aligned}\chi(\mathrm{R}^1 f_\star \mathrm{T}_{X/S}) &= \mathrm{h}^0(\mathrm{R}^1 f_\star \mathrm{T}_{X/S}) - \mathrm{h}^1(\mathrm{R}^1 f_\star \mathrm{T}_{X/S}) = \\ &= \mathrm{h}^0(\mathrm{R}^1 f_\star \mathrm{T}_{X/S}) + \cancel{\mathrm{h}^1(f_\star \mathrm{T}_{X/S})} - \mathrm{h}^1(\mathrm{R}^1 f_\star \mathrm{T}_{X/S}) = \\ &= \mathrm{h}^1(\mathrm{T}_{X/S}) - \mathrm{h}^2(\mathrm{T}_{X/S}) = -\chi(\mathrm{T}_{X/S})\end{aligned}$$

where we applied Serrano-Arakelov to show that $\mathrm{h}^1(f_\star \mathrm{T}_{X/S}) = 0$, and also for the last equality (i.e., to show that $\mathrm{h}^0(\mathrm{T}_{X/S}) = 0$).

Then, using Lemma 2.2, we have

$$\begin{aligned}\chi(\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)) &= \chi(f_\star N) + \chi(\mathrm{R}^1 f_\star \mathrm{T}_{X/S}) = \\ &= \chi(N) - \chi(\mathrm{T}_{X/S}) = \\ &= \chi(f^\star \mathrm{T}_S) - \chi(\mathrm{T}_X) = \\ &= \chi(\mathcal{O}_X) + 2(b-1)(g-1) + (10\chi(\mathcal{O}_X) - 2K_X^2),\end{aligned}$$

where in the last equality we applied Riemann-Roch. □

2.2 *Inequalities for moving curves*

Assume $S = \mathbb{P}^1$, i.e., f is a non-isotrivial rational fibration. In this case, $\chi(\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)) = 11\chi(\mathcal{O}_X) - 2K_X^1 - 2(g-1)$ is the virtual dimension of the deformation problem.

2.5 definition. A non-isotrivial rational fibration is called *free* is $\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)$ is globally generated.

Since we are working with rational base, we have a splitting

$$\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X) \cong \bigoplus_{i=1}^{3g-3} \mathcal{O}(a_i);$$

therefore, being free is equivalent to $a_i \geq 0$ for every i.

2.6 remark. Take $f_\star\, \mathcal{H}\mathrm{om}(\bullet, \mathcal{O}_x)$ of the exact sequence (1); we get

$$f_\star \mathrm{T}_X \to \mathrm{T}_{\mathbb{P}^1} \to \mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X),$$

but the first term is zero by Serrano-Arakelov, hence $\mathrm{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$ injects in $\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)$.

2.7 proposition. *Assume that f is free. Then* Def_f *is smooth of dimension at least* $3g-1$.

Proof. By Remark 2.6, $\mathcal{O}_{\mathbb{P}^1}(2) \subseteq \mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)$, so at least one of the a_i is at least 2. Hence,

$$\dim \mathrm{Def}_f = \mathrm{h}^0(\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)) \geq (3g-4) + 3 = 3g-1. \qquad \square$$

2.8 theorem. *Let $C \subseteq Y$ be a curve with $r := \dim |C| \geq 1$, $\Lambda \subseteq |C|$ a pencil, and $f\colon X \to \mathbb{P}^1$ be the corresponding fibration. If f is free, then*

$$11\chi(\mathcal{O}_Y) - 2K_Y^2 + 2C^2 \geq 5(g-1) + \mathrm{h}^0(\mathcal{O}_Y(C)). \tag{2}$$

Moreover, if $\mathrm{h}^0(K_Y - C) = 0$ (i.e., if C is not contained in the canonical linear system), the inequality becomes

$$10\chi(\mathcal{O}_Y) - 2K_Y^2 \geq 4(g-1) - C^2. \tag{3}$$

Recall that $10\chi(\mathcal{O}_Y) - 2K_Y^2$ (the left hand side of the last inequality) is the virtual dimension of the moduli of the surface Y; so one way to explain this inequality is that in order to have a rational fibration, the moduli space of the surface has to be large enough.

Proof. The linear system Λ has C^2 base points; let X be the blow up of Y at those point; Proposition 2.4 implies

$$11\chi(\mathcal{O}_X) - 2K_X^2 - 2(g-1) = \bigoplus \mathrm{h}^0(\mathcal{O}(a_i)).$$

But $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$ (the characteristic is a birational invariant) and $K_X^2 = K_Y^2 - C^2$ (the canonical self-intersection decrease by one with each blow up), so we obtain

$$11\chi(\mathcal{O}_Y) - 2K_Y^2 + 2C^2 - 2(g-1) = \bigoplus \mathrm{h}^0(\mathcal{O}(a_i)).$$

We know that $\bigoplus \mathrm{h}^0(\mathcal{O}(a_i)) \geq 3g - 1$, but we need a better estimate.

Identify $\mathbb{P}^r = |C|$, so that $\zeta\colon \Lambda = \mathbb{P}^1 \hookrightarrow \mathbb{P}^r = |C|$. This morphism induces the diagram

$$\begin{tikzcd} \mathrm{T}_{\mathbb{P}^1} \arrow[r, hook] \arrow[d, hook] & \mathrm{Ext}^1_f(\Omega_{X/S}, \mathcal{O}_X). \\ \zeta^\star \mathrm{T}_{\mathbb{P}^r} \arrow[r, equal] & \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r+1} \arrow[u, dashed] \end{tikzcd}$$

Our goal is to extend the vertical injection, finding the dashed arrow. The situation is

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathcal{X} \arrow[r, hook] \arrow[d, "F"] & \mathbb{P}^r \times Y \\ \mathbb{P}^1 \arrow[r, "\zeta"'] & \mathbb{P}^r & \end{tikzcd}$$

so we have exact sequences

$$
\begin{aligned}
&0 \to F^\star\Omega^1_{\mathbb{P}^r} \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/\mathbb{P}^r} \to 0,\\
&F_\star \mathrm{T}_{\mathcal{X}} \to \mathrm{T}_{\mathbb{P}^r} \to \mathcal{E}\mathrm{xt}^1_F(\Omega_{\mathcal{X}/\mathbb{P}^r}, \mathcal{O}_{\mathcal{X}}), \text{ and}\\
&0 = f_\star \mathrm{T}_X \to \mathrm{T}_{\mathbb{P}^1} \to \mathcal{E}\mathrm{xt}^1_f(\Omega_{X/\mathbb{P}^1}, \mathcal{O}_X);
\end{aligned}
$$

if we apply $\zeta^\star$ to the second row, then we have vertical maps to the third row, but the first and the third of these map are generically isomorphisms, so $\zeta^\star F_\star \mathrm{T}_{\mathcal{X}}$ is generically zero and $\zeta^\star \mathrm{T}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{r-1}$ injects in $\mathcal{E}\mathrm{xt}^1_f(\Omega_{X/S}, \mathcal{O}_X)$. Therefore,

$$
\begin{aligned}
\bigoplus \mathrm{h}^0(\mathcal{O}(a_i)) &\geq 3 + 2(r-1) + (3g - 3 - r) = \\
&= 3(g-1) + r + 1 = 3(g-1) + \mathrm{h}^0(\mathcal{O}_Y(C)),
\end{aligned}
$$

that is the inequality we need to prove (2).

To deduce the second inequality, we use Riemann-Roch on Y, with the hypothesis $\mathrm{h}^0(K_Y - C) = \mathrm{h}^2(\mathcal{O}_Y(C)) = 0$:

$$
\begin{aligned}
\mathrm{h}^0(\mathcal{O}_Y(C)) &\geq \chi(\mathcal{O}_Y(C)) = \\
&= \chi(\mathcal{O}_Y) + \frac{1}{2}(C^2 - C \cdot K_Y) = \\
&= \chi(\mathcal{O}_Y) - (g-1)) - C^2.
\end{aligned}
$$

$\square$

2.9 theorem. *Let $C \subseteq Y$ be a general curve such that* $\dim |C| \geq 1$ *and Y is non-ruled. Then the fibration $f \colon X \to \mathbb{P}^1$ defined by a pencil $\Lambda \subseteq |C|$ is free. Therefore, the result of the previous theorem holds.*

Proof. We can construct the diagram

$$
\begin{tikzcd}
C \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[r, hook] \arrow[d, "\Phi"'] & \mathcal{Y} \arrow[dl, "\beta"] \\
\mathrm{Spec}(\mathbb{C}) \arrow[r] & V \arrow[r, "\psi_\Phi"'] & \overline{M}_g
\end{tikzcd}
$$

and we can assume V is reduced (actually, we may assume it is smooth).

Let $\mathcal{L} := \mathcal{O}_{\mathcal{Y}}(\mathcal{C})$ and $\mathcal{E} := \beta_\star\mathcal{L}$ (note that $\mathcal{E}$ is locally free of rank $r+1$). Actually, we may assume $r = 1$, i.e. $\mathrm{rk}\,\mathcal{E} = 2$. If this is not the case, we can pullback the whole diagram via $\mathbb{G}(2, \mathcal{E}) \to V$, the Grassmannian of rank 2 sub-bundles of $\mathcal{E}$, which has a canonical rank 2 vector bundle. Also, up to shrinking V, we can assume $\mathcal{E} = \mathcal{O}_{\mathcal{Y}}^{\oplus 2}$, i.e., that $\mathcal{E}$ is trivial.

Blowing up the indeterminacy locus of the pencil $\mathcal{Y}$, we get a morphism $F \colon \mathcal{X} \to \mathbb{P}^1 \times V$; over $\mathbb{P}^1 \times \{v\}$ consider also the fiber X. Now we have the exact sequence

$$
0 \to F^\star\Omega^1_{\mathbb{P}^1 \times V} \to \Omega^1_{\mathcal{X}} \to \Omega^1_{\mathcal{X}/\mathbb{P}^1 \times V} \to 0;
$$

this induces a morphism $K\colon \mathrm{T}_{\mathbb{P}^1\times V} \to \mathcal{E}\mathrm{xt}^1_F(\Omega^1_{\mathcal{X}/\mathbb{P}^1\times V}, \mathcal{O}_{\mathcal{X}})$. This morphism is the Kodaira-Spencer map.

Let $(p,v) \in \mathbb{P}^1 \times V$; then we have

$$\mathrm{T}_{(p,v)}(\mathbb{P}^1\times V) \to \mathcal{E}\mathrm{xt}^1_F(\Omega^1_{\mathcal{X}/\mathbb{P}^1\times V}, \mathcal{O}_{\mathcal{X}})_{(p,v)} \otimes k(p) \to \mathrm{Ext}^1(\Omega^1_{X/(p,v)}, \mathcal{O}_X)$$

and this map is generically surjective.

Restrict now F to $\mathbb{P}^1 \times \{v\}$. We have a splitting

$$\mathrm{T}_{\mathbb{P}^1\times V}|_{\mathbb{P}^1\times\{v\}} = \mathrm{T}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus \dim V}$$

so that this sheaf is globally generated on $\mathbb{P}^1$. Consider

$$\mathcal{E}\mathrm{xt}^1_F(\Omega^1_{\mathcal{X}/\mathbb{P}^1\times V}, \mathcal{O}_{\mathcal{X}}) \otimes \mathcal{O}_{\mathbb{P}^1\times\{v\}} \to \mathcal{E}\mathrm{xt}^1_f(\Omega^1_{X/\mathbb{P}^1}, \mathcal{O}_X);$$

it is generically surjective, so the latter is generated somewhere, so being on $\mathbb{P}^1$, this implies that it is globally generated. $\square$

2.10 example. We have $10\chi - 2K^2 \geq 4(g-1) - C^2$ if $\mathrm{h}^0(K-C) = 0$. For $Y = \mathbb{P}^2$, the assumption is true, so we consider a nonsingular degree d curve $C \subseteq \mathbb{P}^2$. The inequality becomes $-8 \geq 2d(d-3) - d^2 = d^2 - 6d$, i.e. $d \leq 4$. This recover the classical result that state that for a plane curve to have general moduli, the degree has to be at most 4.

2.3 *Bounding the genera of curves*

Lecture 4 (1 hour) September 3rd, 2010

The hope is to use the inequalities of Theorem 2.8 to bound from above the genus of curves with determined properties.

2.11 example. Let Y be a K3 surface, $C \subseteq Y$; then $\dim|C| = g$, so, if we deform the curve and the surface together, we get at most $19 + g$ parameters. Hence $3g - 3 \leq 19 + g$, that is $g \leq 11$. Instead, if we apply Theorem 2.8, we get that, since $\chi(\mathcal{O}_Y) = 2$ and $K_Y^2 = 0$,

$$20 \geq 4(g-1) - 2(g-1),$$

that is $g \leq 11$ again.

2.12 proposition. *Let Y be a surface with $\kappa(Y) \geq 0$; let $C \subseteq Y$ be a general curve of genus g, moving in a linear system of dimension at least* 1*; then, if* $\mathrm{h}^0(K_Y - C) = 0$, *we have $g \leq 6 + 5p_g$.*

Note that for surfaces with geometric genus equals zero, the condition $\mathrm{h}^0(K_Y - C) = 0$ is automatically satisfied, so we cannot find movable curves of high genera in these surfaces.

Proof. Let Z be the minimal model of Y and $\sigma\colon Y \to Z$ the corresponding morphism, factored as

$$Y = Z_\delta \xrightarrow{\sigma_\delta} Z_{\delta-1} \to \cdots \to Z_1 \xrightarrow{\sigma_1} Z.$$

Consider D_i, the image of C in Z_i. We can assume that the center of σ_i is the contraction of a singular point of D_{i-1}.

Now, $K_Y^2 = K_Z^2 - \delta$, so

$$\begin{aligned} C^2 &= 2(g-1) - C \cdot K_Y = \\ &= 2(g-1) - C \cdot (\sigma^\star K_Z + 2r_i E_i) \leq \\ &\leq 2(g-1) - (\sigma(C) \cdot K_Z + 2\delta) \leq 2(g-1) - 2\delta. \end{aligned}$$

Combining this inequality with Theorem 2.8, we get

$$\begin{aligned} 10\chi(\mathcal{O}_Z) - 2K_Z^2 - 2\delta &= 10\chi(\mathcal{O}_Y) - 2K_Y^2 \geq \\ &\geq 4(g-1) - C^2 \geq 4(g-1) - (2(g-1) - 2\delta) = \\ &= 2(g-1) - 2\delta, \end{aligned}$$

hence $10\chi(\mathcal{O}_Z) - 2K_Z^2 \geq 2(g-1)$. But $K_Z^2 \geq 0$, so $10\chi(\mathcal{O}_Z) \geq 2(g-1)$, and we conclude using the fact that $\chi(\mathcal{O}_Z) \leq p_g + 1$. □

2.13 example (Bruno-Verra). The inequalities of Theorem 2.8 are quite sharp. For example, there is a curve $C \subseteq \mathbb{P}^6$ of genus 15 and degree 19 that lies in a surface $Y = Q_1 \cap \cdots \cap Q_4$ that is the intersection of four quadrics. The curves moves in Y in a net, and we have $C^2 = 9$, $K_Y^2 = 16$, $\chi(\mathcal{O}_Y) = 8$. The inequality becomes

$$48 = 10\chi - 2K_Y^2 > 4(g-1) - C^2 = 47.$$

Using some more time, we could prove that for elliptic surfaces we can the genus of moving curves is at most 16. We devote instead the rest of the lecture to the same estimate for surface of general type.

2.4 *Bounding for surfaces of general type*

2.14 theorem. *Let Y be a surface of general type (not necessarily minimal), Z the minimal model, $C \subseteq Y$ a general curve of genus g. Assume* $\dim |C| \geq 2$ *and* $K_Z^2 \geq 3\chi(\mathcal{O}_Z) - 10$*; then* $g \leq 19$*.*

The inequality in the assumption is the Castelnuovo condition, which is frequently seen in surface theory. Instead, there is an unfortunate additional request: the curve needs to move in a net instead of in a pencil.

Before proving the theorem we need to do a digression. Let C be a genus g curve. The *Clifford index* of C is defined as

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(L) \mid \mathrm{h}^0(L), \mathrm{h}^1(L) \geq 2,$$

where $\mathrm{Cliff}(L) := \deg L - 2\,\mathrm{h}^0(L) + 2$. It is clear that $\mathrm{Cliff}(C) \geq 0$, but the property we need is that if C is general, then $\mathrm{Cliff}(C) = [g - 1/2]$.

Proof of the theorem. We can assume the same factorization of $\sigma\colon Y \to Z$ we did before, with the same notations and properties. Let $L := \mathcal{O}_C(\sigma^\star K_Z)$; then, since $K_Y = \sigma^\star K_Z + \sum r_i E_i$,

$$\begin{aligned} \mathrm{h}^0(\omega_C L^{-1}) &= \mathrm{h}^0(\mathcal{O}_C(C + K_Y - \sigma^\star K_Z)) = \\ &= \mathrm{h}^0(\mathcal{O}_C(C + \textstyle\sum r_i E_i)) \geq \mathrm{h}^0(\mathcal{O}_C(C)) \geq 2, \end{aligned}$$

by the assumption on the linear system $|C|$.

To use Theorem 2.8, we need to prove $\mathrm{h}^0(K_Y - C) = 0$. If this wasn't true, $2C < K_Y + C$, so $\mathcal{O}_C(2C) \subseteq \omega_C$. But this is impossible because the curve is general, hence twice his line bundle cannot be special.

Moreover, $\sigma^\star K_Z - C \subseteq K_Y - C$, so that $\mathrm{h}^0(\sigma^\star K_Z - C) \leq \mathrm{h}^0(K_Y - C) = 0$. Using the sequence

$$0 \to \sigma^\star K_Z - C \to \sigma^\star K_Z \to L \to 0,$$

we prove that $\mathrm{h}^0(L) \geq \mathrm{h}^0(\sigma^\star K_Z) = p_g \geq 2$.

Therefore, the line bundle L contributes to $\mathrm{Cliff}(C)$ and we get the following results:

$[g-1/2] = \mathrm{Cliff}(C) \leq \deg L - 2p_g + 2$	since L contributes to $\mathrm{Cliff}(C)$;
$10\chi(\mathcal{O}_Z) - 2K_Z^2 + 2\delta \geq 4(g-1) - C^2$	by Theorem 2.8;
$4\chi(\mathcal{O}_Z) + 20 + 2\delta \geq 4(g-1) - C^2$	using Castelnuovo condition;
$4(1+p_g) + 20 + 2\delta \geq 4(g-1) - C^2$	because $1 + p_g \geq \chi$;
$28 + 2\delta \geq 4(g-1) - C^2 - 4p_g + 4$	transforming the previous one;
$2(g-1) - C^2 \geq \deg L + 2\delta$	since the former is $\deg \mathcal{O}_C(K_Y)$;
$28 + 2\delta \geq 2(\deg L + 2\delta - 2p_g + 2) + C^2$	substituting the previous one;
$28 + 2\delta \geq g - 2 + 4\delta + C^2$	including the estimate on Cliff;
$C^2 \geq 1/2 g + 1$	since $\mathcal{O}_C(C)$ moves in a pencil.

Using this last inequality, we get $28 - 2\delta \geq 3/2 g + 1$, that is $g \leq 19$. □

2.15 remark. We have not explored the case $K_Z^2 < 3\chi(\mathcal{O}_Z) - 10$. In principle, this case is easier, since such surfaces admit a double cover over a rational surface, and we can reduce the problem to the rational case. In this case, we will have to assume that $\dim |C|$ is at least 1 or 2 depending on how we will manage to solve the rational case. The problem is that the rational case seems not so easy.

We finish stating one last result.

2.16 theorem. *Let $C \subseteq \mathbb{P}^2$ be a nodal curve, whose normalization is general of genus g. Assume that* $\dim |C| \geq 1$. *Then $g \leq 9$.*

Blowing up the node, instead of having a singular curve moving in a fixed surface, we have a smooth curve deforming together with a surface.