

DERIVED DEFORMATION FUNCTORS

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CONTENTS

1	Introduction	1
2	Motivation	2
2.1	Intersection theory	2
2.2	Cotangent complex	2
2.3	Obstruction theory	3
2.4	DGLA	4
3	Homotopy theory	8
3.1	Model categories	8
3.2	Path objects	9
3.3	Homotopy function objects	9
3.4	MC again	10
3.5	Homotopy fiber products	11
3.6	Quillen functors	11
4	Derived deformation theory	12
4.1	Functor categories	12
4.2	Simplicial functors	14

1 INTRODUCTION

Classical deformation theory studies functors defined on Artinian rings. Opposed to this, derived deformation theory studies functors on simplicial rings, or dg-rings.

A simplicial rings looks like

$$A_0 \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} A_1 \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} A_2 \quad \cdots$$

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2. MOTIVATION

Over \mathbb{Q} , they are equivalent to dg-algebras (on non-negative chain degrees).

1.1 DEFINITION. A *dg-algebra* A is a chain complex

$$\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow \cdots$$

together with a multiplication $A_i \times A_j \rightarrow A_{i+j}$ such that $ab = (-1)^{\deg a \deg b} ba$, $d(ab) = da \cdot b + (-1)^{\deg a} a \cdot db$, $1 \in A_0$.

As a convention, we will write dg-algebra when we mean a chain complex, and DG-algebra when we mean a cochain complex.

Let \mathfrak{dgAlg} be the category of dg-algebras and $\mathfrak{dg}_+\mathfrak{Alg}$ the category of dg-algebras with non-negative degrees.

1.2 DEFINITION. A *local dg-algebra* A is an Artinian dg-algebra such that:

1. $\dim A < \infty$;
2. $\mathfrak{m}(A)^n = 0$ for n high enough, where $\mathfrak{m}(A) := \ker(A \rightarrow k)$ (remember that $A = k \oplus \mathfrak{m}(A)$).

We will write \mathfrak{dgArt} and $\mathfrak{dg}_+\mathfrak{Art}$ for the categories of all local dg-Artinian rings and with non-negative degrees.

2 MOTIVATION

2.1 Intersection theory

Consider $\{0\} \in \mathbb{A}^1$ and $X := \{0\} \times_{\mathbb{A}^1}^h \{0\}$, where \times^h is the homotopic fiber product; let $X = \mathrm{Spec}(k \otimes_{k[t]}^{\mathbb{L}} k)$, where $\otimes^{\mathbb{L}}$ is the derived tensor product. Now,

$$k \cong (k[t] \cdot s \xrightarrow{d} k[t]) =: A$$

(here $ds = t$ and $k[t]$ is in level 0). Then again $X = \mathrm{Spec}(A \otimes_{k[t]} k) = \mathrm{Spec}(k \oplus k[-1])$.

The multiplicity is the Euler characteristic, that is $\chi(X) = 1 - 1 = 0$. Let $x := \{0\}$; then $T_x X = k[-1]$ and $\dim X = \chi(k[-1]) = -1$.

2.2 Cotangent complex

Let $B \rightarrow R$ be a non-smooth morphism of rings; take a quasi free resolution $\tilde{R}_\bullet \rightarrow R$ over B , that means that \tilde{R}_\bullet is free as a graded B -algebra and $\tilde{R}_\bullet \rightarrow R$ is a quasi isomorphism (i.e., an isomorphism on H^*).

2.1 DEFINITION. The *cotangent complex* is defined as

$$\mathbb{L}_\bullet^{R/B} := \Omega(\tilde{R}_\bullet/B) \otimes_{\tilde{R}_\bullet} R.$$

Note that if $J = \ker(\tilde{R}_\bullet \otimes_B R \rightarrow R)$, then $\mathbb{L}_\bullet^{R/B} = I/J^2$.

Let us review some properties of the cotangent complex. If we have a square zero extension $A \rightarrow B$ with kernel I , then the obstruction to lifting the flat morphism $B \rightarrow R$ to a flat morphism $A \rightarrow R$ with $R \otimes_A B \cong R$ is $\text{Ext}_R^2(\mathbb{L}_\bullet^{R/B}, R \otimes_B I)$. This is in turn the second cohomology of the complex

$$\text{Hom}_R(\mathbb{L}_0, R \otimes_B I) \rightarrow \text{Hom}_R(\mathbb{L}_1, R \otimes_B I) \rightarrow \cdots$$

If R does lift, the set of isomorphism classes of lifts is isomorphic to $\text{Ext}_R^1(\mathbb{L}_\bullet^{R/B}, R \otimes_B I)$. We can choose \tilde{R}_\bullet canonically, so the construction sheafify.

2.2 EXAMPLE. Let $X \hookrightarrow Y$ be a regular embedding over S with ideal I ; assume that Y is smooth. Then $\mathbb{L}_\bullet^{X/S}$ is isomorphic to $(j^* \Omega_{Y/S} \leftarrow I/I^2)$ (here the degree are 0 and 1). Moreover, $\text{Ext}_X^*(\mathbb{L}_\bullet^{X/S}, \mathcal{O}_X \otimes I)$ governs global deformations.

2.3 Obstruction theory

Take a nice functor $F: \mathfrak{Art} \rightarrow \mathfrak{Sets}$; consider a semismall extension $A \rightarrow B$ with kernel I , and let $x \in F(B)$. What is the obstruction to lifting x to $F(A)$?

Assume that F extends to a functor from \mathfrak{dgArt} .

2.3 DEFINITION. We say that $F: \mathfrak{dgArt} \rightarrow \mathfrak{Sets}$ is a *deformation functor* if:

1. for every $A \rightarrow B$ semismall with kernel I (i.e., with $I \cdot \mathfrak{m}(A) = 0$), and for every $C \rightarrow B$, we have that

$$F(A \times_B C) \rightarrow F(A) \times_{F(B)} F(C)$$

is surjective;

2. for every $A, B \in \mathfrak{dgArt}$, $F(A \times_k B) \rightarrow F(A) \times F(B)$ is an isomorphism;
3. $F(k) = \{\text{pt}\}$;
4. if $f: A \rightarrow B$ is an acyclic semismall extension (i.e., $H_*(\ker f) = 0$), then $F(A) \rightarrow F(B)$ is an isomorphism.

Fix now a semismall extension $A \rightarrow B$ with kernel I , in the category \mathfrak{Art} .

2.4 DEFINITION. Consider ε_n on level n , with $\varepsilon_n^2 = 0$; define $H^n F := F(k[\varepsilon_n])$; these are k -vector spaces.

Let \tilde{B} be a dg-algebra, with A at level 0 and I at level 1; the morphism $I \hookrightarrow A$ is the kernel morphism. Then we have a obvious morphism of dg-algebras $\tilde{B} \rightarrow B$, whose kernel is $I \rightarrow I$.

So $\tilde{B} \rightarrow B$ is an acyclic semismall extension in \mathfrak{dgArt} ; by the fourth property, $F(\tilde{B}) \cong F(B)$, so there is an $\tilde{x} \in F(\tilde{B})$ associated to $x \in F(B)$.

2. MOTIVATION

Now, $\tilde{B} \rightarrow k \oplus I[-1]$, so we have a dg-morphism

$$\begin{array}{ccc} I & \longrightarrow & I \\ \downarrow & & \downarrow 0 \\ A & \longrightarrow & k \end{array}$$

Observe that $A = \tilde{B} \times_{(k \oplus I[-1])} k$. By the first property, $F(A) \rightarrow F(\tilde{B}) \times_{F(k \oplus I[-1])} F(k)$. Now, $F(k)$ is a point, and $F(k \oplus I[-1]) = (H^1 F) \otimes I$. So

$$F(A) \rightarrow F(B) \times_{H^1 F \otimes I} \{0\}$$

is a surjection and $x \in F(B)$ lifts to $F(A)$ if and only if the map $F(B) \rightarrow H^1 F \otimes I$ sends x to 0; hence $H^1(F)$ is the obstruction theory.

In the following we will give some examples of situations where these functors arise.

2.4 DGLA

2.5 DEFINITION. A *differential graded Lie algebra*, or *DGLA*, is a cochain complex equipped with a bracket operator $[\cdot, \cdot]: L^i \times L^j \rightarrow L^{i+j}$, satisfying:

1. $[b, a] = -(-1)^{\deg a \deg b} [a, b]$;
2. $[[a, b], c] = [a, [b, c]] - (-1)^{\deg a \deg b} [b, [a, c]]$;
3. $d[a, b] = [da, b] + (-1)^{\deg a} [a, db]$.

2.6 EXAMPLE. The typical example of a DGLA is when I is a Lie algebra, A^\bullet is the de Rham complex; then $A^\bullet \otimes I$ is a DGLA.

2.7 DEFINITION. We define the *Maurer-Cartan functor* $\text{MC}(L): \mathfrak{dg}\mathfrak{Art} \rightarrow \mathfrak{Sets}$ as the functor that sends $A \in \mathfrak{dg}\mathfrak{Art}$ to the set

$$\left\{ \omega \in \prod_n L^{n+1} \otimes \mathfrak{m}(A)_n \mid d\omega + 1/2[\omega, \omega] = 0 \right\}.$$

2.8 DEFINITION. The *gauge group* $\text{Gg}(L): \mathfrak{Art} \rightarrow \mathfrak{Groups}$ is defined by

$$\text{Gg}(L, A) := \exp\left(\prod_n L^n \otimes \mathfrak{m}(A)_n\right).$$

The gauge group acts on $\text{MC}(L, A)$ by $(g, \omega) \mapsto g\omega g^{-1} - dg \cdot g^{-1}$.

2.9 DEFINITION. We define

$$\text{Def}(L, A) := \text{MC}(L, A) / \text{Gg}(L, A).$$

This can be proven to be a deformation functor.

2.10 EXAMPLE. Take a dg-resolution $\tilde{R}_\bullet \rightarrow R$ over the base B , and let $L^n := \text{Der}_B(\tilde{R}_\bullet, \tilde{R}_\bullet)^n$, i.e. the derivations of the form $\tilde{R}_\bullet \rightarrow \tilde{R}_\bullet[-n]$ over B , considered as a graded ring. We have $[f, g] = f \circ g \mp g \circ f$ and $df = d \circ f \mp f \circ d$.

In this situation, $\text{MC}(L, A)$ is the set of deformations δ of d on $\tilde{R}_\bullet \otimes A$ such that $\delta^2 = 0$, and $\delta(rs) = \delta r \circ s \pm r \circ \delta s$. This is equivalent to say that $(\tilde{R}_\bullet \otimes A, \delta)$, where $\delta = d + \omega$, is a dg-algebra over A . Also, the gauge group $\text{Gg}(L, A)$ is the set of infinitesimal automorphisms of $\tilde{R}_\bullet \otimes A$ as a graded algebra. Finally, $\text{Def}(L, A)$ is the set of isomorphism classes of $\tilde{R}_\bullet \otimes A$.

If $A \in \mathfrak{A}rt$, this maps to the set of deformations of $\tilde{R}_\bullet \otimes A$ by

$$(\tilde{R}_\bullet \otimes A, \delta) \mapsto H_0(\tilde{R}_\bullet \otimes A, \delta)$$

(the target is the set of deformations of R). Also, $H^1 L = \text{Ext}_R^1(\mathbb{L}_\bullet^{R/B}, R)$ and $H^i(\text{Def } L) = H^{i+1} L$. This last assertion is true in general.

Lecture 2 (1 hour)
September 1st, 2010

2.11 EXAMPLE. Let X be a scheme over k , and \mathcal{F} be an \mathcal{O}_X -module. Consider an injective resolution \mathcal{I}^\bullet of \mathcal{F} , and define $L^\bullet := \text{END}_{\mathcal{O}_X}^\bullet(\mathcal{I}^\bullet)$, so that we have $L^n = \text{Hom}_{\mathcal{O}_X}(\mathcal{I}^\bullet, \mathcal{I}^\bullet[n])$. Define also

$$\begin{aligned} [f, g] &:= f \circ g - (-1)^{\deg f \deg g} g \circ f, \text{ and} \\ df &= d \circ f - (-1)^{\deg f} f \circ d. \end{aligned}$$

Let $A \in \mathfrak{A}rt_k$, $\omega \in \text{MC}(L, A)$; then

$$d + \omega: \mathcal{I}^n \otimes A \rightarrow \mathcal{I}^{n+1} \otimes A \rightarrow \dots$$

and the deformations of \mathcal{F} are $\mathcal{H}^0(\mathcal{I}^\bullet \otimes A, d + \omega)$. In general, $\text{MC}(L)$ determines $\text{Def}(L)$.

2.12 THEOREM (Manetti). *The functor $\text{Def}(L): \mathfrak{d}g\mathfrak{A}rt \rightarrow \mathfrak{S}ets$ is the universal deformation functor under $\text{MC}(L)$, i.e., for any deformation functor F with a transformation $\text{MC}(L) \rightarrow F$, there exists a unique compatible transformation $\text{Def}(L) \rightarrow F$.*

We can wonder what other deformation functors are there.

2.13 THEOREM. *If $F: \mathfrak{d}g\mathfrak{A}rt \rightarrow \mathfrak{S}ets$ is a deformation functor, there exists a DGLA L such that $\text{Def}(L) \cong F$.*

In fact, Def determines a functor from DGLAs to deformation functors that induces an equivalence between $H_0(\mathfrak{D}Gla)$, the homotopy category obtained by formally inverting quasi isomorphisms, and deformation functors.

2.14 PROBLEM. The functor $\text{Def}(L)$ is not left exact (i.e., does not preserve fiber products). In particular, it cannot sheafify, so it does not admit a global version.

The solution to this problem, given by Hinich, is to look at functors to simplicial sets.

2. MOTIVATION

2.15 DEFINITION. The *topological n -simplex* $|\Delta^n| \subseteq \mathbb{R}^{n+1}$ is given by the set of tuples (x_0, \dots, x_n) with $\sum x_i = 1$. There are maps $\partial^i: |\Delta^{n-1}| \rightarrow |\Delta^n|$ for $0 \leq i \leq n$ called the *i -th face* and maps $\sigma^i: |\Delta^{n+1}| \rightarrow |\Delta^n|$ collapsing the edge between v_i and v_{i+1} called *collapsing maps*.

2.16 DEFINITION. Given $X \in \mathfrak{Top}$, define $\text{Sing}(X)$ to be the diagram

$$\begin{array}{ccccc} & & \partial_2 & & \\ & & \curvearrowright & & \\ & \partial_1 & & \sigma_0 & \\ \text{Sing}(X)_0 & \xleftarrow{\quad} & \text{Sing}(X)_1 & \xrightarrow{\quad} & \text{Sing}(X)_2 & \dots \\ & \sigma_0 & & \partial_1 & \\ & \curvearrowleft & & \curvearrowright & \\ & \partial_0 & & \sigma_1 & \\ & & \partial_0 & & \end{array}$$

where $\text{Sing}(X)_n = \text{Hom}(|\Delta^n|, X)$. Any diagram like this is called a *simplicial set*. Denote the category of these by \mathfrak{S} .

The functor $\text{Sing}: \mathfrak{Top} \rightarrow \mathfrak{S}$ has a left adjoint $K \mapsto |K|$, that is, we have isomorphisms

$$\text{Hom}_{\mathfrak{Top}}(|K|, X) \cong \text{Hom}_{\mathfrak{S}}(K, \text{Sing}(X)).$$

2.17 REMARK. Dold-Kahn gives an equivalence between simplicial abelian groups and non-negative chain complexes.

2.18 DEFINITION. Given $K \in \mathfrak{S}$, define $\pi_0 K := \pi_0 |K|$, and for $x \in \pi_0 K$, $\pi_1(K, x) := \pi_1(|K|, x)$.

The canonical maps $|\text{Sing}(X)| \rightarrow X$ and $K \rightarrow \text{Sing}(|K|)$ are weak equivalences, i.e., they give isomorphisms on π_n .

2.19 DEFINITION. Let x_i be at level 0; then we define

$$\Omega_{\text{dR}}^\bullet(|\Delta^n|) := \frac{\mathbb{Q}[x_1, \dots, x_n, dx_1, \dots, dx_n]}{\sum x_i = 1, \sum dx_i = 0}.$$

2.20 DEFINITION. Given a DGLA L , define $\text{MC}(\underline{L}): \text{dg}\mathfrak{A} \rightarrow \mathfrak{S}$ by

$$\text{MC}(\underline{L}, A)_n := \text{MC}(L \otimes \Omega_{\text{dR}}^\bullet(|\Delta^n|), A).$$

The idea is that $\pi_0 \text{MC}(\underline{L}, A)$ will be the set of (quasi) isomorphism classes of objects, and $\pi_1(\text{MC}(\underline{L}, A), x)$ will be the automorphisms group of x , and $\pi_n(\text{MC}(\underline{L}, A), x)$ will be the higher automorphism groups. In high-brow language, \mathfrak{S} is a model for ∞ -groupoids.

Note that if $H^i(L) = 0$ for $i < 0$ and $A \in \mathfrak{A}$ or $A \in \text{dg}\mathfrak{A}$, then $\pi_n(\text{MC}(\underline{L}, A), x) = 0$ for any $i \geq 2$.

These are some properties of $\text{MC}(\underline{L})$.

1. It takes acyclic semismall extensions to weak equivalences.
2. $\pi_0(\text{MC}(\underline{L})) = \text{Def}(L)$.

3. If $\varepsilon_n^2 = 0$ modulo n , then $\pi_i(\mathrm{MC}(\underline{L}, k[\varepsilon_n])) = H^{n+1-i}(L)$.
4. If $H^i(L) = 0$ for every $i < 0$ and $A \in \mathfrak{A}rt$, then $\mathrm{MC}(\underline{L}, A)$ is weakly equivalent to the nerve of the groupoid $[\mathrm{MC}(\underline{L}, A)/G\mathfrak{g}(\underline{L}, A)]$.

2.21 EXAMPLE. When $L = \mathrm{DER}(\tilde{R}_\bullet, \tilde{R}_\bullet)$, where $\tilde{R}_\bullet \rightarrow R$ is a free resolution, then $\mathrm{MC}(\underline{L})_n$ is the Simplicial set of deformations of $\tilde{R}_\bullet \otimes \Omega_{\mathrm{dR}}^\bullet(|\Delta^n|)$.

2.22 PROBLEM. The functor $\mathrm{MC}(\underline{L}): \mathfrak{d}\mathfrak{g}\mathfrak{A}rt \rightarrow \mathfrak{S}$ does not take all quasi isomorphisms to weak equivalences. To solve this problem, we restrict the functor to $\mathfrak{d}\mathfrak{g}_+\mathfrak{A}rt \subseteq \mathfrak{d}\mathfrak{g}\mathfrak{A}rt$; $\mathrm{MC}(\underline{L}): \mathfrak{d}\mathfrak{g}_+\mathfrak{A}rt \rightarrow \mathfrak{S}$ is called the *Hinich's simplicial nerve* of L .

Our next aim is to understand how to recover L from $\mathrm{MC}(\underline{L})$ (restricted to $\mathfrak{d}\mathfrak{g}_+\mathfrak{A}rt$).

2.23 DEFINITION. Let $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ be maps of topological spaces. We define the *homotopy fiber product* to be $X \times_Y^h Z := X \times_Y Y^{[0,1]} \times_Y Z$.

2.24 REMARK. Let $P := X \times_Y^h Z$; then there is a long exact sequence

$$\pi_n(P) \rightarrow \pi_n(X) \times \pi_n(Z) \rightarrow \pi_n(Y) \rightarrow \pi_{n-1}(P) \rightarrow \cdots$$

In particular, $\pi_0(P) \rightarrow \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z)$.

If $A \rightarrow B$ is semismall and $C \rightarrow B$, then

$$\mathrm{MC}(\underline{L}, A \times_B C) \cong \mathrm{MC}(\underline{L}, A) \times_{\mathrm{MC}(\underline{L}, B)}^h \mathrm{MC}(\underline{L}, C).$$

2.25 THEOREM. Let A, B, C be as above and consider the category \mathcal{C} of functors $F: \mathfrak{d}\mathfrak{g}_+\mathfrak{A}rt \rightarrow \mathfrak{S}$ such that:

- $|F(A \times_B C)| \xrightarrow{\sim} |F(A)| \times_{|F(B)|}^h |F(C)|$ is an isomorphism, and
- F takes quasi isomorphism to weak equivalence.

Then the association $L \mapsto \mathrm{MC}(\underline{L}, A)$ gives a functor $\mathfrak{D}\mathfrak{G}\mathfrak{L}\mathfrak{a} \rightarrow \mathcal{C}$, that induces an equivalence between the homotopy categories $H_0(\mathfrak{D}\mathfrak{G}\mathfrak{L}\mathfrak{a})$ and $H_0(\mathcal{C})$. In fact, this is a ∞ -equivalence.

The homotopy category is constructed inverting weak equivalences. For DGLAs this is clear; for \mathcal{C} we say that $F \rightarrow G$ is a weak equivalence if and only if $F(A) \rightarrow G(A)$ is a weak equivalence for every A .

2.26 EXAMPLE. Let V be a vector space, $L^0 := \mathrm{End}(V)$, $L^i := 0$ for every $i \neq 0$. For every $A \in \mathfrak{d}\mathfrak{g}_+\mathfrak{A}rt$, $\pi_0(\mathrm{MC}(\underline{L}, A))$ is the set of isomorphism classes of deformations of $V \otimes A$ as an A -module (complex). Moreover, $\pi_1(\mathrm{MC}(\underline{L}, A))$ is the set of homotopy classes of maps $V \rightarrow V \otimes \mathfrak{m}(A)[n-1]$.

3 HOMOTOPY THEORY

3.1 Model categories

3.1 DEFINITION. A *model category* is a category \mathcal{C} endowed with three classes of distinguished morphisms: the class F of *fibrations*, W of *weak equivalences*, and C of *cofibrations*, subject to the following properties:

1. every map $A \rightarrow B$ has factorizations as $A \xrightarrow{C} X \xrightarrow{W \cap F} B$ and $A \xrightarrow{W \cap C} Y \xrightarrow{F} B$.
2. Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

with $i \in C$ and $p \in F$, of $i, p \in W$, then there exists a diagonal arrow $B \rightarrow X$ commuting with the diagram.

We refer to elements of $W \cap F$ as *trivial fibrations*, and to elements of $W \cap C$ as *trivial cofibrations*.

An object X is *fibrant* if the morphism from X to the final object is a fibration. An object A is *cofibrant* if the morphism from the initial object to A is a cofibration.

The second property allows us to recover C from $W \cap F$ and F from $W \cap C$. Indeed, consider the following.

3.2 DEFINITION. We say that i has the *left lifting property* with respect to p (or that p has the *right lifting property* with respect to i) if in the situation of the second property, a diagonal arrow exists.

We have that $i \in C$ if and only if it has the left lifting property with respect to all $p \in W \cap F$; conversely, $p \in F$ if and only if it has the right lifting property with respect to all $i \in W \cap C$.

3.3 EXAMPLE. These are examples of model categories (we write \star when a class is too long to be described):

- the categories of chain complexes of vector spaces, where the cofibrations are injections, fibrations are surjections and weak equivalences are quasi isomorphisms;
- chain complexes of vector spaces in degree ≥ 0 , with injections, surjections in degree > 0 , and quasi isomorphisms;
- \mathcal{S} , with injections, Kan fibrations, and weak equivalences;
- $\mathcal{T}op$, with \star , Serre fibrations, and weak equivalences;

- chain complexes of sheaves in degree ≥ 0 , with injections in degree > 0 , surjections with injective kernel, and quasi isomorphisms.
- DGLAs, with \star , surjections and quasi isomorphisms.

Lecture 3 (1 hour)
September 2nd, 2010

Recall that given a model category \mathcal{C} , we defined the homotopy category $H_0(\mathcal{C})$ formally inverting weak equivalences. We also write $[A, X]$ for $\text{Hom}_{H_0(\mathcal{C})}(A, X)$.

Note that not all objects in \mathcal{C} are fibrant, but $\text{Sing}(X)$ is, as is any simplicial abelian group. Moreover, for every $K \in \mathcal{C}$, $K \rightarrow \text{Sing}(|K|)$ is a fibrant replacement (that is, is a weak equivalence to a fibrant object).

3.2 Path objects

3.4 DEFINITION. A *path object* for a fibrant object X in a model category \mathcal{C} is a diagram

$$\begin{array}{ccccc} X & \longrightarrow & PX & \rightrightarrows & X \\ & \searrow & & \nearrow & \\ & & \text{id} & & \end{array}$$

with $X \rightarrow PX$ a weak equivalence, and $PX \rightarrow X \times X$ a fibration.

Since X is assumed to be fibrant, also PX is fibrant. The axioms of the model category implies the existence of PX , but not its uniqueness.

3.5 EXAMPLE.

1. In \mathfrak{Top} , $PX = X^{[0,1]}$, the space of maps $[0, 1] \rightarrow X$.
2. In \mathcal{S} , $PX = X^{\Delta^1}$, where, for any object $K \in \mathcal{S}$, $(X^K)_n := \text{Hom}(\Delta^n \times K, X)$.
3. For chain complexes, PV is the cylinder of the map $V \rightarrow V \oplus V$ (the underlying graded ring is $V \oplus V \oplus V[-1]$).
4. For DGLAs, $PL = L \otimes \Omega_{\text{dR}}^\bullet(|\Delta^1|)$.

3.6 THEOREM. If A is cofibrant and X is fibrant, then $[A, X] = \text{Hom}(A, X) / \text{Hom}(A, PX)$.

The dual notion of path objects is the notion of *cylinder objects*.

3.3 Homotopy function objects

Given $X \in \mathcal{C}$ fibrant, we say that a simplicial diagram $\text{RES}(X)$ in \mathcal{C}

$$X = \text{RES}_0(X) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \text{RES}_1(X) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \text{RES}_2(X) \quad \dots$$

is a *fibrant simplicial resolution* if:

1. $X \rightarrow \text{RES}_n(X)$ is a weak equivalence for every n ;

3. HOMOTOPY THEORY

2. $\text{RES}_n(X) \rightarrow M_n \text{RES}(X)$ is a fibration, where M_n is the n -th Ready matching object.

The boundary $\partial\Delta^n$ of Δ^n is the equalizer of the two maps

$$\prod_{0 \leq i < j \leq n} \Delta^{n-2} \rightarrow \prod_{i=0}^n \Delta^{n-1}.$$

We define $M_n \text{RES}(X)$ to be the equalizer of the two maps

$$\prod_{i=0}^{n-1} \text{RES}_{n-1}(X) \rightarrow \prod_{0 \leq i < j \leq n} \text{RES}_{n-2}(X).$$

3.7 DEFINITION. Given $A, X \in \mathcal{C}$, with X fibrant and A cofibrant, we define the *homotopy function complex* as $\text{RMap}(A, X) \in \mathcal{S}$ by

$$\text{RMap}(A, X)_n := \text{Hom}(A, \text{RES}_n(X)).$$

If X and A are general, we define $\text{RMap}(A, X)$ as $\text{RMap}(A', \hat{X})$, where \hat{X} is a fibrant replacement for X and A' is a cofibrant replacement for A . The result is independent on the choice of the replacements.

3.8 THEOREM (Dwyer-Kan). *The construction of RMap depends only on $W \subseteq \mathcal{C}$.*

3.9 EXAMPLE.

1. In $\mathfrak{dg}_+ \mathfrak{Vect}$, $\text{RES}_n(V) = V \otimes Nk(\Delta^n)$ is generated by non-degenerate i -simplexes in Δ^n in level i .
2. In \mathcal{S} , $\text{RES}_n(X) = X^{\Delta^n}$.
3. For DGLAs, $\text{RES}_n(L) = L \otimes \Omega_{\text{dR}}^\bullet(|\Delta^n|)$.

Note that $\text{RES}_1(X)$ is always a path object.

3.4 MC again

The following is called *cobar construction*. Given $A \in \mathfrak{dgAlgt}$, let $\beta^*(A)$ be the free graded Lie algebra on generators $\mathfrak{m}(A)^\vee[-1]$. The differential is given by $d_A + \Delta: \mathfrak{m}(A)^\vee \rightarrow \mathfrak{m}(A)^\vee[-1] \oplus \mathfrak{m}(A)^\vee[-2]$, where Δ is the dual of the multiplication.

These are the main properties of the cobar construction:

1. $\beta^*(A)$ is cofibrant;
2. $\text{MC}(L, A) = \text{Hom}_{\mathfrak{DGLa}}(\beta^*(A), L)$;
3. $\text{Def}(L, A) = [\beta^*(A), L]$;
4. $\text{MC}(\underline{L}, A) = \text{RMap}(\beta^*A, L)$.

3.5 Homotopy fiber products

3.10 DEFINITION. Given $A \rightarrow B$ and $C \rightarrow B$ with B fibrant, define the *homotopy fiber product* $A \times_B^h C$ as $A' \times_B C'$, where $A \rightarrow A'$ is a weak equivalence, $A' \rightarrow B$ is a fibration, and similarly for C' .

If \mathcal{C} is right proper (as is every category we have seen), this is equivalent to $A' \times_B C$, that is, we don't need to replace C with C' .

3.11 EXAMPLE. The object $A \times_B PB$ is a replacement for A , fibrant over B . As a corollary, $A \times_B^h C \simeq A \times_B PB \times_B C$.

3.6 Quillen functors

3.12 DEFINITION. Given two adjoint functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ (that is, we have $\text{Hom}(FA, B) \cong \text{Hom}(A, GB)$), we say that F is *left Quillen* or that G is *right Quillen* if either:

1. F preserve cofibrations and trivial cofibrations, or
2. G preserve fibrations and trivial fibrations.

3.13 DEFINITION. Given a right Quillen functor $G: \mathcal{D} \rightarrow \mathcal{C}$, we define the *right derived functor* RG by $RG(X) := G(X')$, where $X \rightarrow X'$ is a fibrant replacement. We define LF for F left Quillen dually.

3.14 EXAMPLE. Let $f: X \rightarrow Y$ be a map of topological spaces; then we have functors $f_*: \mathfrak{dg}_+ \mathfrak{Sh}(X) \rightarrow \mathfrak{dg}_+ \mathfrak{Sh}(Y)$ and f^* the other way round. All objects are cofibrant, so $Lf^* = f^*$; but $Rf_*(V)$ is $f_* I^\bullet$ for $V \rightarrow I^\bullet$ a fibrant replacement, i.e., an injective resolution.

These are some properties of Quillen functors:

1. $\text{RMap}(LFA, B) \simeq \text{RMap}(A, RGB)$;
2. $LF: H_0(\mathcal{C}) \rightarrow H_0(\mathcal{D})$ and $RG: H_0(\mathcal{D}) \rightarrow H_0(\mathcal{C})$ are well defined.

3.15 DEFINITION. We say that (F, G) are a *Quillen equivalence* if LF and RG are an equivalence between $H_0(\mathcal{C})$ and $H_0(\mathcal{D})$.

If (F, G) are a Quillen equivalence, $\text{RMap}(LFA, LFA') \simeq \text{RMap}(A, A')$ and similarly for RG .

3.16 EXAMPLE.

1. The functor $\text{Sing}: \mathfrak{Top} \rightarrow \mathfrak{S}$ and $|\bullet|: \mathfrak{S} \rightarrow \mathfrak{Top}$ form a Quillen equivalence.
2. The cotangent complex $\mathbb{L}_{R/B}$ can be describes as the left Quillen functor $\mathfrak{dg}_+ \mathfrak{Alg}_B \downarrow_R \rightarrow \mathfrak{dg} \mathfrak{Mod}_R$.

4 DERIVED DEFORMATION THEORY

4.1 Functor categories

We observe that $\mathfrak{dg}\mathfrak{Art}$ is too small to be a model category. The solution is the following.

4.1 DEFINITION. We say that $F: \mathcal{C} \rightarrow \mathcal{D}$ is *left exact* if

$$F(A \times_B C) \cong F(A) \times_{F(B)} F(C)$$

and it preserves the final object. Let $\text{lex}(\mathcal{C}, \mathcal{D})$ be the category of left exact functor $\mathcal{C} \rightarrow \mathcal{D}$.

Consider $\text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets})$; this contains $(\mathfrak{dg}\mathfrak{Art})^{\text{op}}$ as a full subcategory. To prove this, we can associate to an object $A \in \mathfrak{dg}\mathfrak{Art}$ the functor $A \mapsto \text{Spec } A$, where $(\text{Spec } A)(B) := \text{Hom}(A, B)$.

4.2 THEOREM. *There is a model structure on $\text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets})$ with these properties:*

1. *all objects are cofibrant;*
2. *F is fibrant (trivially fibrant) if and only if $F(A) \rightarrow F(B)$ is surjective for all acyclic semismall extensions (all semismall extensions);*
3. *a map of fibrant objects $F \rightarrow G$ is a weak equivalence if and only if the map of universal deformation functors $F^+ \rightarrow G^+$ is an isomorphisms.*

Note that, with this definition of model structure, $\text{MC}(L)$ is fibrant, while $\text{Gg}(L)$ is trivially fibrant.

4.3 LEMMA. *The object $\text{MC}(L) \xrightarrow{s} \text{MC}(L) \times \text{Gg}(L) \xrightarrow{t_1, t_2} \text{MC}(L)$ is a path object.*

Proof. Since $\text{Gg}(L)$ is trivially fibrant, s is a weak equivalence. Then $\text{MC}(L) \times \text{Gg}(L) \rightarrow \text{MC}(L) \times \text{MC}(L)$ is a fibration. This says that for every $A \twoheadrightarrow B$ semismall and acyclic, and for every $x, y \in \text{MC}(L, A)$ and $g \in \text{Gg}(L, B)$, such that $g(\bar{x}) = \bar{y} \in \text{MC}(L, B)$, there exists $\tilde{g} \in \text{Gg}(L, A)$ over g such that $\tilde{g}(x) = y$. \square

4.4 REMARK. The functors $F \in \text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets})$ are precisely $\text{MC}(V)$, where V is a L_∞ -algebra.

4.5 THEOREM. *Given a deformation functor $F: \mathfrak{dg}\mathfrak{Art} \rightarrow \mathfrak{Sets}$, there exists a fibrant $G \in \text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets})$ such that $F(A) = [\text{Spec } A, G]$ for every $A \in \mathfrak{dg}\mathfrak{Art}$. This association induces an equivalence between the category of deformation functors and the homotopy category $\text{H}_0(\text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets}))$.*

Outline of the proof. The functor F can be extended to inverse systems setting

$$F(\{A_i\}_{i \in I}) := \varprojlim_{i \in I} F(A_i).$$

By Grothendieck prorepresentability, one proves that objects of $\text{lex}(\mathfrak{dgAlgt}, \mathfrak{Sets})$ are $\varinjlim_{i \in I} \text{Spec } A_i$. Hence, F becomes a functor $\text{lex}(\mathfrak{dgAlgt}, \mathfrak{Sets})^{\text{op}} \rightarrow \mathfrak{Sets}$.

Now, we apply Heller's theorem, that states the existence of G such that $[\varinjlim_{i \in I} \text{Spec } A_i, G] = F(\{A_i\}_{i \in I})$ for every $\{A_i\}_{i \in I}$. \square

4.6 THEOREM. *The functor $\text{MC}: \mathfrak{DGLa} \rightarrow \text{lex}(\mathfrak{dgAlgt}, \mathfrak{Sets})$ is a right Quillen equivalence.*

Proof. Given $\varinjlim_{i \in I} \text{Spec } A_i$, we get a DGLA $\varinjlim_{i \in I} \beta^*(A_i)$ using the cobar construction. Then

$$\text{Hom}(\varinjlim_{i \in I} (\beta^*(A_i)), L) = \varprojlim_{i \in I} \text{Hom}(\beta^*(A_i), L) = \varprojlim_{i \in I} \text{MC}(L, A_i).$$

So β^* is a left adjoint to MC ; but MC preserves fibrations and trivial fibrations, hence is right Quillen.

The last part of the proof involves spectral sequences. \square

Given a DGLA L , let $\beta(A)$ be the inverse system $k[L[1]^\vee]$ (note that $L[i]$ is a pro-finite dimensional vector space). Then, $\text{MC}(L) = \text{Specf } \beta(L)$, and the differential are given on the generators by $d_L + \Delta: L[i]^\vee \rightarrow L[2]^\vee \oplus \wedge^2 L[2]^\vee$, where Δ is the dual to $[\bullet, \bullet]$.

Consider the space $\text{Hom}(\text{MC}(L), \text{MC}(M))$; from what we said, this is equal to $\text{Hom}(\text{Specf } \beta(L), \text{MC}(M))$, hence it is

$$\text{MC}(M, \beta(L)) = \text{Hom}_{\mathfrak{DGLa}}(\beta^* \beta(L), M).$$

Infact, we see that this space is the space of ∞ -maps $L \rightarrow M$. Therefore, $\text{RMap}(L, M) = \text{MC}(\underline{M}, \beta(L))$. Applying Fiorenza-Martinengo, this gives the Griffiths period map, Bogomolov-Tian-Todorov, the Kodaira embedding principle, Goldman-Millson and other theorems.

Right Quillen functors preserve homotopy fiber products of DGLAs; therefore, given $\chi: L \rightarrow M$, we have

$$\text{MC}(L \times_M^h \{0\}) = \text{MC}(L) \times_{\text{MC}(M)}^h \{0\}.$$

Also, $\text{MC}(M) \times \text{Gg}(M)$ is a path object for $\text{MC}(M)$, so

$$\text{MC}(L \times_M^h \{0\}) = (\text{MC}(L) \times \text{Gg}(M)) \times_{\text{MC}(M)} \{0\}.$$

Then, Manetti-Fiorenza shows that there is a L_∞ -algebra C_χ with $\text{MC}(C_\chi) = (\text{MC}(L) \times \text{Gg}(M)) \times_{\text{MC}(M)} \{0\}$.

In the general framework, this translates to the fact that

$$L \times_M (M \otimes \Omega_{\text{dR}}^\bullet(\Delta^1)) \times_M \{0\}$$

has the same properties (but with quasi isomorphisms).

4.2 Simplicial functors

Function complexes in $\text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets})$. Take the pullback $A(n)'$ of $k \rightarrow \Omega_{\text{dR}}^\bullet(|\Delta^n|)$ via $A \otimes \Omega_{\text{dR}}^\bullet(|\Delta^n|) \rightarrow \Omega_{\text{dR}}^\bullet(|\Delta^n|)$. In general this is not finite dimensional. So we consider $I := \ker(A(n)' \rightarrow A^{\times \kappa(n+1)})$ and do the following.

4.7 DEFINITION. We define $A(n) := \{A(n)'/I^r\}_r$, and $\text{RES}_n(F)(A) := \varprojlim_r F(A(n)'/I^r)$.

$\text{RES}_n(F)(A)$ is a simplicial fibrant resolution. Write $\underline{F}: \mathfrak{dg}\mathfrak{Art} \rightarrow \mathfrak{S}$ for the functor defined by $\underline{F}_n := \text{RES}_n(F)$.

4.8 REMARK. It is not true that $\text{MC}(\underline{L}) = \underline{\text{MC}}(L)$. However, we have a weak equivalence $\text{MC}(\underline{L}) \hookrightarrow \underline{\text{MC}}(L)$ defined by

$$L \otimes \Omega_{\text{dR}}^\bullet(|\Delta^n|) \otimes A \rightarrow \varprojlim_r (L \otimes \Omega_{\text{dR}}^\bullet(|\Delta^n|) \otimes A) / I^r.$$

Moreover,

$$\text{MC}(\underline{L}, A) \simeq \text{RMap}(\beta^*(A), L) \simeq \text{RMap}(\text{Spec } A, \text{MC}(L)) \simeq \underline{\text{MC}}(L)(A).$$

4.9 REMARK. All objects in $\mathfrak{D}\mathfrak{G}\mathfrak{a}$ are fibrant, so $\text{RMC} = \text{MC}$; likewise, all objects in $\text{lex}(\mathfrak{dg}\mathfrak{Art}, \mathfrak{Sets})$ are cofibrant, so $L\beta^* = \beta^*$.

4.10 THEOREM. *There is a model structure on $\text{lex}(\mathfrak{dg}_+\mathfrak{Art}, \mathfrak{S})$ for which all objects are cofibrant and F is fibrant if and only if*

1. for all semismall extensions $A \twoheadrightarrow B$, $F(A) \rightarrow F(B)$ is a fibration;
2. for all acyclic semismall extensions $A \twoheadrightarrow B$, $F(A) \rightarrow F(B)$ is a trivial fibration.

A transformation $\eta: F \rightarrow G$ of fibrant objects is a weak equivalence if and only if $F(A) \rightarrow G(A)$ is a weak equivalence for every object $A \in \mathfrak{dg}_+\mathfrak{Art}$.

4.11 THEOREM. *Let \mathcal{S} be the category of functors $F: \mathfrak{dg}_+\mathfrak{Art} \rightarrow \mathfrak{S}$ such that:*

1. for every semismall extension $A \twoheadrightarrow B$, and for every $C \rightarrow B$, the map

$$F(A \times_B C) \rightarrow F(A) \times_{F(B)}^h F(C)$$

is a weak equivalence;

2. F preserves weak equivalences;
3. $F(k)$ is constructible.

Then, the map $\text{lex}(\mathfrak{dg}_+\mathfrak{Art}, \mathfrak{S}) \rightarrow \mathcal{S}$, given by sending F to a fibrant replacement, is an ∞ -equivalence. In particular, $H_0(\text{lex}(\mathfrak{dg}_+\mathfrak{Art}, \mathfrak{S})) \simeq H_0(\mathfrak{S})$

4.12 REMARK. An ∞ -equivalence is a transformation that induces an equivalence on the homotopy categories plus a condition on how space of maps translates under this transformation. More precisely, we have to impose the condition $\text{RMap}(X, Y) = \mathcal{H}\text{om}(X, Y)$, where $\mathcal{H}\text{om}(X, Y)_n := \text{Hom}(X, Y^{\Delta^n})$.

4.13 THEOREM. *The functor $\text{lex}(\partial\mathfrak{g}\mathfrak{A}rt, \mathfrak{S}ets) \rightarrow \text{lex}(\partial\mathfrak{g}_+\mathfrak{A}rt, \$)$ sending F to the restriction $\underline{F}|_{\partial\mathfrak{g}_+\mathfrak{A}rt}$ is a right Quillen equivalence.*

Proof. We look at RMap , after constructing a left adjoint. □

So the idea is that $\text{lex}(\partial\mathfrak{g}_+\mathfrak{A}rt, \$)$ is the category of simplicial formal dg-schemes, and corresponds to the category of cosimplicial pro-Artinian dg-rings. Also, $\text{lex}(\partial\mathfrak{g}\mathfrak{A}rt, \mathfrak{S}ets)$ corresponds to pro-Artinian dg-rings. The functor $\text{lex}(\partial\mathfrak{g}_+\mathfrak{A}rt, \$) \rightarrow \text{lex}(\partial\mathfrak{g}\mathfrak{A}rt, \mathfrak{S}ets)$ is Thom-Whitney.

Summing up, we obtained the following:

$$\begin{array}{ccc}
 & \mathfrak{D}\mathfrak{G}la & \text{lex}(\mathfrak{s}\mathfrak{A}rt, \$) \\
 \text{MC} \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) \beta^* & & \uparrow N^*F \\
 \text{lex}(\partial\mathfrak{g}\mathfrak{A}rt, \mathfrak{S}ets) & \xleftarrow{\text{Thom}} & \text{lex}(\partial\mathfrak{g}_+\mathfrak{A}rt, \$) \\
 \uparrow & \xrightarrow{F \mapsto \underline{F}} & \\
 L_\infty\text{-algebras} & &
 \end{array}$$

The functor $N: \partial\mathfrak{g}_+\mathfrak{A}rt \rightarrow \mathfrak{s}\mathfrak{A}rt$ is a right Quillen functor induced by the Dold-Kan normalization, and induces N^* in the diagram.

Given a dg-manifold X , the associated functor $\partial\mathfrak{g}_+\mathfrak{A}lg \rightarrow \$$ is \underline{X} , given by $\underline{X}(A)_n := \text{Hom}(\text{Spec}(A \otimes \Omega_{dR}^\bullet(\Delta^n)), X)$. This is a 0-truncated geometric stack (as in Toën-Vezzosi) or a derived 0-stack (as in Lurie).