Derived deformation functors

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Contents

1 Introduction

Lecture 1 (1 hour)
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Classical deformation theory studies functors defined on Artinian rings. Opposed to this, derived deformation theory studies functors on simplicial rings, or dg-rings.

A simplicial rings looks like

$$A_0 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; A_1 \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow}\; A_2 \qquad \cdots$$

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Over $\mathbb{Q}$, they are equivalent to dg-algebras (on non-negative chain degrees).

1.1 definition. A *dg-algebra* A is a chain complex

$$\cdots \to A_1 \to A_0 \to \cdots$$

together with a multiplication $A_i \times A_j \to A_{i+j}$ such that $ab = (-1)^{\deg a \deg b} ba$, $d(ab) = da \cdot b + (-1)^{\deg a} a \cdot db$, $1 \in A_0$.

As a convention, we will write dg-algebra when we mean a chain complex, and DG-algebra when we mean a cochain complex.

Let $\mathfrak{dgAlg}$ be the category of dg-algebras and $\mathfrak{dg}_+\mathfrak{Alg}$ che category of dg-algebras with non-negative degrees.

1.2 definition. A *local dg-algebra* A is an Artinian dg-algebra such that:

1. $\dim A < \infty$;
2. $\mathfrak{m}(A)^n = 0$ for n high enough, where $\mathfrak{m}(A) := \ker(A \to k)$ (remember that $A = k \oplus \mathfrak{m}(A)$).

We will write $\mathfrak{dgArt}$ and $\mathfrak{dg}_+\mathfrak{Art}$ for the categories of all local dg-Artinian rings and with non-negative degrees.

2 Motivation

2.1 *Intersection theory*

Consider $\{0\} \in \mathbb{A}^1$ and $X := \{0\} \times^h_{\mathbb{A}^1} \{0\}$, where $\times^h$ is the homotopic fiber product; let $X = \operatorname{Spec}(k \otimes^{\mathbb{L}}_{k[t]} k)$, where $\otimes^{\mathbb{L}}$ is the derived tensor product. Now,

$$k \cong (k[t] \cdot s \xrightarrow{d} k[t]) =: A$$

(here $ds = t$ and $k[t]$ is in level 0). Then again $X = \operatorname{Spec}(A \otimes_{k[t]} k) = \operatorname{Spec}(k \oplus k[-1])$.

The multiplicity is the Euler characteristic, that is $\chi(X) = 1 - 1 = 0$. Let $x := \{0\}$; then $\mathrm{T}_x X = k[-1]$ and $\dim X = \chi(k[-1]) = -1$.

2.2 *Cotangent complex*

Let $B \to R$ be a non-smooth morphism of rings; take a quasi free resolution $\widetilde{R}_\bullet \to R$ over B, that means that $\widetilde{R}_\bullet$ is free as a graded B-algebra and $\widetilde{R}_\bullet \to R$ is a quasi isomorphism (i.e., an isomorphism on $\mathrm{H}^\star$).

2.1 definition. The *cotangent complex* is defined as

$$\mathbb{L}^{R/B}_\bullet := \Omega({\widetilde{R}_\bullet}/{B}) \otimes_{\widetilde{R}_\bullet} R.$$

Note that if $J = \ker(\widetilde{R}_\bullet \otimes_B R \to R)$, then $\mathbb{L}_\bullet^{R/B} = J/J^2$.

Let us review some properties of the cotangent complex. If we have a square zero extension $A \to B$ with kernel I, then the obstruction to lifting the flat morphism $B \to R$ to a flat morphism $A \to R$ with $R \otimes_A B \cong R$ is $\operatorname{Ext}^2_R(\mathbb{L}_\bullet^{R/B}, R \otimes_B I)$. This is in turn the second cohomology of the complex

$$\operatorname{Hom}_R(\mathbb{L}_0, R \otimes_B I) \to \operatorname{Hom}_R(\mathbb{L}_1, R \otimes_B I) \to \cdots.$$

If R does lift, the set of isomorphism classes of lifts is isomorphic to $\operatorname{Ext}^1_R(\mathbb{L}_\bullet^{R/B}, R \otimes_B I)$. We can choose $\widetilde{R}_\bullet$ canonically, so the construction sheafify.

2.2 example. Let $X \hookrightarrow Y$ be a regular embedding over S with ideal I; assume that Y is smooth. Then $\mathbb{L}_\bullet^{X/S}$ is isomorphic to $(j^\star \Omega_{Y/S} \leftarrow I/I^2)$ (here the degree are 0 and 1). Moreover, $\operatorname{Ext}^\star_X(\mathbb{L}_\bullet^{X/S}, \mathcal{O}_X \otimes I)$ governs global deformations.

2.3 *Obstruction theory*

Take a nice functor $F\colon \mathfrak{Art} \to \mathfrak{Sets}$; consider a semismall extension $A \to B$ with kernel I, and let $x \in F(B)$. What is the obstruction to lifting x to $F(A)$?

Assume that F extends to a functor from $\mathfrak{dgArt}$.

2.3 definition. We say that $F\colon \mathfrak{dgArt} \to \mathfrak{Sets}$ is a *deformation functor* if:

1. for every $A \twoheadrightarrow B$ semismall with kernel I (i.e., with $I \cdot \mathfrak{m}(A) = 0$), and for every $C \to B$, we have that

   $$F(A \times_B C) \to F(A) \times_{F(B)} F(C)$$

   is surjective;

2. for every $A, B \in \mathfrak{dgArt}$, $F(A \times_k B) \to F(A) \times F(B)$ is an isomorphism;

3. $F(k) = \{\mathrm{pt}\}$;

4. if $f\colon A \twoheadrightarrow B$ is an acyclic semismall extension (i.e., $\mathrm{H}_\star(\ker f) = 0$), then $F(A) \to F(B)$ is an isomorphism.

Fix now a semismall extension $A \twoheadrightarrow B$ with kernel I, in the category $\mathfrak{Art}$.

2.4 definition. Consider ε_n on level n, with $\varepsilon_n^2 = 0$; define $\mathrm{H}^n F := F(k[\varepsilon_n])$; these are k-vector spaces.

Let $\widetilde{B}$ be a dg-algebra, with A at level 0 and I at level 1; the morphism $I \hookrightarrow A$ is the kernel morphism. Then we have a obvious morphism of dg-algebras $\widetilde{B} \to B$, whose kernel is $I \to I$.

So $\widetilde{B} \to B$ is an acyclic semismall extension in $\mathfrak{dgArt}$; by the fourth property, $F(\widetilde{B}) \cong F(B)$, so there is an $\widetilde{x} \in F(\widetilde{B})$ associated to $x \in F(B)$.

Now, $\widetilde{B} \twoheadrightarrow k \oplus I[-1]$, so we have a dg-morphism

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] & I \arrow[d, "0"] \\ A \arrow[r] & k \end{tikzcd}$$

Observe that $A = \widetilde{B} \times_{(k \oplus I[-1])} k$. By the first property, $F(A) \twoheadrightarrow F(\widetilde{B}) \times_{F(k \oplus I[-1])} F(k)$. Now, $F(k)$ is a point, and $F(k \oplus I[-1]) = (\mathrm{H}^1 F) \otimes I$. So

$$F(A) \twoheadrightarrow F(B) \times_{\mathrm{H}^1 F \otimes I} \{0\}$$

is a surjection and $x \in F(B)$ lifts to $F(A)$ if and only if the map $F(B) \to \mathrm{H}^1 F \otimes I$ sends x to 0; hence $\mathrm{H}^1(F)$ is the obstruction theory.

In the following we will give same examples of situations where these functors arise.

2.4 *DGLA*

2.5 definition. A *differential graded Lie algebra*, or *DGLA*, is a cochain complex equipped with a bracket operator $[,]\colon L^i \times L^j \to L^{i+j}$, satisfying:

1. $[b, a] = -(-1)^{\deg a \deg b}[a, b]$;
2. $[[a, b], c] = [a, [b, c]] - (-1)^{\deg a \deg b}[b, [a, c]]$;
3. $d[a, b] = [da, b] + (-1)^{\deg a}[a, db]$.

2.6 example. The typical example of a DGLA is when I is a Lie algebra, $A^\bullet$ is the de Rham complex; then $A^\bullet \otimes I$ is a DGLA.

2.7 definition. We define the *Maurer-Cartan functor* $\mathrm{MC}(L)\colon \mathfrak{dgArt} \to \mathfrak{Sets}$ as the functor that sends $A \in \mathfrak{dgArt}$ to the set

$$\left\{ w \in \prod_n L^{n+1} \otimes \mathfrak{m}(A)_n \;\middle|\; d\omega + 1/2[\omega, \omega] = 0 \right\}.$$

2.8 definition. The *gauge group* $\mathrm{Gg}(L)\colon \mathfrak{Art} \to \mathfrak{Groups}$ is defined by

$$\mathrm{Gg}(L, A) := \exp\Big(\prod_n L^n \otimes \mathfrak{m}(A)_n\Big).$$

The gauge group acts on $\mathrm{MC}(L, A)$ by $(g, \omega) \mapsto g\omega g^{-1} - dg \cdot g^{-1}$.

2.9 definition. We define

$$\mathrm{Def}(L, A) := {}^{\mathrm{MC}(L, A)}\!\big/_{\mathrm{Gg}(L, A)}.$$

This can be proven to be a deformation functor.

2.10 EXAMPLE. Take a dg-resolution $\widetilde{R}_\bullet \to R$ over the base B, and let $L^n := \mathrm{Der}_B(\widetilde{R}_\bullet, \widetilde{R}_\bullet)^n$, i.e. the derivations of the form $\widetilde{R}_\bullet \to \widetilde{R}_\bullet[-n]$ over B, considered as a graded ring. We have $[f,g] = f \circ g \mp g \circ f$ and $df = d \circ f \mp f \circ d$.

In this situation, $\mathrm{MC}(L, A)$ is the set of deformations δ of d on $\widetilde{R}_\bullet \otimes A$ such that $\delta^2 = 0$, and $\delta(rs) = \delta r \circ s \pm r \circ \delta s$. This is equivalent to say that $(\widetilde{R}_\bullet \otimes A, \delta)$, where $\delta = d + \omega$, is a dg-algebra over A. Also, the gauge group $\mathrm{Gg}(L, A)$ is the set of infinitesimal automorphisms of $\widetilde{R}_\bullet \otimes A$ as a graded algebra. Finally, $\mathrm{Def}(L, A)$ is the set of isomorphism classes of $\widetilde{R}_\bullet \otimes A$.

If $A \in \mathfrak{Art}$, this maps to the set of deformations of $\widetilde{R}_\bullet \otimes A$ by

$$(\widetilde{R}_\bullet \otimes A, \delta) \mapsto \mathrm{H}_0(\widetilde{R}_\bullet \otimes A, \delta)$$

(the target is the set of deformations of R). Also, $\mathrm{H}^1 L = \mathrm{Ext}^1_R(\mathbb{L}_\bullet^{R/B}, R)$ and $\mathrm{H}^i(\mathrm{Def}\, L) = \mathrm{H}^{i+1} L$. This last assertion is true in general.

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2.11 EXAMPLE. Let X be a scheme over k, and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Consider an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}$, and define $L^\bullet := \mathrm{END}^\bullet_{\mathcal{O}_X}(\mathcal{I}^\bullet)$, so that we have $L^n = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}^\bullet, \mathcal{I}^\bullet[n])$. Define also

$$\begin{aligned} [f,g] &:= f \circ g - (-1)^{\deg f \deg g} g \circ f, \text{ and} \\ df &= d \circ f - (-1)^{\deg f} f \circ d. \end{aligned}$$

Let $A \in \mathfrak{Art}_k$, $\omega \in \mathrm{MC}(L, A)$; then

$$d + \omega \colon \mathcal{I}^n \otimes A \to \mathcal{I}^{n+1} \otimes A \to \cdots$$

and the deformations of $\mathcal{F}$ are $\mathcal{H}^0(\mathcal{I}^\bullet \otimes A, d + \omega)$. In general, $\mathrm{MC}(L)$ determines $\mathrm{Def}(L)$.

2.12 THEOREM (Manetti). *The functor* $\mathrm{Def}(L)\colon \mathfrak{dgArt} \to \mathfrak{Sets}$ *is the universal deformation functor under* $\mathrm{MC}(L)$*, i.e., for any deformation functor F with a transformation* $\mathrm{MC}(L) \to F$*, there exists a unique compatible transformation* $\mathrm{Def}(L) \to F$*.*

We can wonder what other deformation functors are there.

2.13 THEOREM. *If $F\colon \mathfrak{dgArt} \to \mathfrak{Sets}$ is a deformation functor, there exists a DGLA L such that* $\mathrm{Def}(L) \cong F$*.*

Infact, Def determines a functor from DGLAs to deformation functors that induces an equivalence between $\mathrm{H}_0(\mathfrak{DGla})$, the homotopy category obtained by formally inverting quasi isomorphisms, and deformation functors.

2.14 PROBLEM. The functor $\mathrm{Def}(L)$ is not left exact (i.e., does not preserve fiber products). In particular, it cannot sheafify, so it does not admit a global version.

The solution to this problem, given by Hinich, is to look at functors to simplicial sets.

2.15 DEFINITION. The *topological n-simplex* $|\Delta^n| \subseteq \mathbb{R}^{n+1}$ is given by the set of tuples $(x_0, \dots, x_n)$ with $\sum x_i = 1$. There are maps $\partial^i \colon |\Delta^{n-1}| \to |\Delta^n|$ for $0 \leq i \leq n$ called the *i-th face* and maps $\sigma^i \colon |\Delta^{n+1}| \to |\Delta^n|$ collapsing the edge between v_i and v_{i+1} called *collapsing maps*.

2.16 DEFINITION. Given $X \in \mathfrak{Top}$, define $\mathrm{Sing}(X)$ to be the diagram

$$\begin{tikzcd}
\mathrm{Sing}(X)_0 \arrow[r, "\sigma_0"] & \mathrm{Sing}(X)_1 \arrow[l, bend right, "\partial_1"'] \arrow[l, bend left, "\partial_0"] \arrow[r, bend left, "\sigma_0"] \arrow[r, bend right, "\sigma_1"'] & \mathrm{Sing}(X)_2 \arrow[l, "\partial_1"] \arrow[l, bend right=50, "\partial_2"'] \arrow[l, bend left=50, "\partial_0"] & \cdots
\end{tikzcd}$$

where $\mathrm{Sing}(X)_n = \mathrm{Hom}(|\Delta^n|, X)$. Any diagram like this is called a *simplicial set*. Denote the category of these by $\$$.

The functor $\mathrm{Sing} \colon \mathfrak{Top} \to \$$ has a left adjoint $K \mapsto |K|$, that is, we have isomorphisms

$$\mathrm{Hom}_{\mathfrak{Top}}(|K|, X) \cong \mathrm{Hom}_{\$}(K, \mathrm{Sing}(X)).$$

2.17 REMARK. Dold-Kahn gives an equivalence between simplicial abelian groups and non-negative chain complexes.

2.18 DEFINITION. Given $K \in \$$, define $\pi_0 K := \pi_0 |K|$, and for $x \in \pi_0 K$, $\pi_1(K, x) := \pi_1(|K|, x)$.

The canonical maps $|\mathrm{Sing}(X)| \to X$ and $K \to \mathrm{Sing}(|K|)$ are weak equivalences, i.e., they give isomorphisms on π_n.

2.19 DEFINITION. Let x_i be at level 0; then we define

$$\Omega^\bullet_{\mathrm{dR}}(|\Delta^n|) := \frac{\mathbb{Q}[x_1, \dots, x_n, \mathrm{d}\, x_1, \dots, \mathrm{d}\, x_n]}{\sum x_i = 1, \sum \mathrm{d}\, x_i = 1}.$$

2.20 DEFINITION. Given a DGLA L, define $\mathrm{MC}(\underline{L}) \colon \mathfrak{dgArt} \to \$$ by

$$\mathrm{MC}(\underline{L}, A)_n := \mathrm{MC}(L \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|), A).$$

The idea is that $\pi_0 \mathrm{MC}(\underline{L}, A)$ will be the set of (quasi) isomorphism classes of objects, and $\pi_1(\mathrm{MC}(\underline{L}, A), x)$ will be the automorphisms group of x, and $\pi_n(\mathrm{MC}(\underline{L}, A), x)$ will be the higher automorphism groups. In high-brow language, $\$$ is a model for ∞-groupoids.

Note that if $\mathrm{H}^i(L) = 0$ for $i < 0$ and $A \in \mathfrak{Art}$ or $A \in \mathfrak{dgArt}$, then $\pi_n(\mathrm{MC}(\underline{L}, A), x) = 0$ for any $i \geq 2$.

These are some properties of $\mathrm{MC}(\underline{L})$.

1. It takes acyclic semismall extensions to weak equivalences.

2. $\pi_0(\mathrm{MC}(\underline{L})) = \mathrm{Def}(L)$.

3. If $\varepsilon_n^2 = 0$ modulo n, then $\pi_i(\mathrm{MC}(\underline{L}, k[\varepsilon_n])) = \mathrm{H}^{n+1-i}(L)$.

4. If $\mathrm{H}^i(L) = 0$ for every $i < 0$ and $A \in \mathfrak{Art}$, then $\mathrm{MC}(\underline{L}, A)$ is weakly equivalent to the nerve of the groupoid $\left[{}^{\mathrm{MC}(L,A)}/_{\mathrm{Gg}(L,A)}\right]$.

2.21 EXAMPLE. When $L = \mathrm{DER}(\widetilde{R}_\bullet, \widetilde{R}_\bullet)$, where $\widetilde{R}_\bullet \to R$ is a free resolution, then $\mathrm{MC}(\underline{L})_n$ is the Simplicial set of deformations of $\widetilde{R}_\bullet \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|)$.

2.22 PROBLEM. The functor $\mathrm{MC}(\underline{L})\colon \mathfrak{dgArt} \to \$$ does not take all quasi isomorphisms to weak equivalences. To solve this problem, we restrict the functor to $\mathfrak{dg}_+\mathfrak{Art} \subseteq \mathfrak{dgArt}$; $\mathrm{MC}(\underline{L})\colon \mathfrak{dg}_+\mathfrak{Art} \to \$$ is called the *Hinich's simplicial nerve* of L.

Our next aim is to understand how to recover L from $\mathrm{MC}(\underline{L})$ (restricted to $\mathfrak{dg}_+\mathfrak{Art}$).

2.23 DEFINITION. Let $f\colon X \to Y$ and $g\colon Z \to Y$ be maps of topological spaces. We define the *homotopy fiber product* to be $X \times^h_Y Z := X \times_Y Y^{[0,1]} \times_Y Z$.

2.24 REMARK. Let $P := X \times^h_Y Z$; then there is a long exact sequence

$$\pi_n(P) \to \pi_n(X) \times \pi_n(Z) \to \pi_n(Y) \to \pi_{n-1}(P) \to \cdots$$

In particular, $\pi_0(P) \twoheadrightarrow \pi_0(X) \times_{\pi_0(Y)} \pi_0(Z)$.

If $A \twoheadrightarrow B$ is semismall and $C \to B$, then

$$\mathrm{MC}(\underline{L}, A \times_B C) \cong \mathrm{MC}(\underline{L}, A) \times^h_{\mathrm{MC}(\underline{L},B)} \mathrm{MC}(\underline{L}, C).$$

2.25 THEOREM. *Let A, B, C be as above and consider the category $\mathcal{C}$ of functors $F\colon \mathfrak{dg}_+\mathfrak{Art} \to \$$ such that:*

- $|F(A \times_B C)| \xrightarrow{\sim} |F(A)| \times^h_{|F(B)|} |F(C)|$ *is an isomorphism, and*

- *F takes quasi isomorphism to weak equivalence.*

Then the association $L \mapsto \mathrm{MC}(\underline{L}, A)$ gives a functor $\mathfrak{DGla} \to \mathcal{C}$, that induces an equivalence between the homotopy categories $\mathrm{H}_0(\mathfrak{DGla})$ and $\mathrm{H}_0(\mathcal{C})$. Infact, this is a ∞-equivalence.

The homotopy category is constructed inverting weak equivalences. For DGLAs this is clear; for $\mathcal{C}$ we say that $F \to G$ is a weak equivalence if and only if $F(A) \to G(A)$ is a weak equivalence for every A.

2.26 EXAMPLE. Let V be a vector space, $L^0 := \mathrm{End}(V)$, $L^i := 0$ for every $i \neq 0$. For every $A \in \mathfrak{dg}_+\mathfrak{Art}$, $\pi_0(\mathrm{MC}(\underline{L}, A))$ is the set of isomorphism classes of deformations of $V \otimes A$ as an A-module (complex). Moreover, $\pi_1(\mathrm{MC}(\underline{L}, A))$ is the set of homotopy classes of maps $V \to V \otimes \mathfrak{m}(A)[n-1]$.

3 Homotopy theory

3.1 *Model categories*

3.1 definition. A *model category* is a category $\mathcal{C}$ endowed with three classes of distinguished morphisms: the class F of *fibrations*, W of *weak equivalences*, and C of *cofibrations*, subject to the following properties:

1. every map $A \to B$ has factorizations as $A \xrightarrow{C} X \xrightarrow{W \cap F} B$ and $A \xrightarrow{W \cap C} Y \xrightarrow{F} B$.
2. Given a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

   with $i \in C$ and $p \in F$, of $i, p \in W$, then there exists a diagonal arrow $B \to X$ commuting with the diagram.

We refer to elements of $W \cap F$ as *trivial fibrations*, and to elements of $W \cap C$ as *trivial cofibrations*.

An object X is *fibrant* if the morphism from X to the final object is a fibration. An object A is *cofibrant* if the morphism from the initial object to A is a cofibration.

The second property allows us to recover C from $W \cap F$ and F from $W \cap C$. Indeed, consider the following.

3.2 definition. We say that i has the *left lifting property* with respect to p (or that p has the *right lifting property* with respect to i) if in the situation of the second property, a diagonal arrow exists.

We have that $i \in C$ if and only if it has the left lifting property with respect to all $p \in W \cap F$; conversely, $p \in F$ if and only if it has the right lifting property with respect to all $i \in W \cap C$.

3.3 example. These are examples of model categories (we write $\star$ when a class is too long to be described):

- the categories of chain complexes of vector spaces, where the cofibrations are injections, fibrations are surjections and weak equivalences are quasi isomorphisms;
- chain complexes of vector spaces in degree ≥ 0, with injections, surjections in degree > 0, and quasi isomorphisms;
- $\$$, with injections, Kan fibrations, and weak equivalences;
- $\mathfrak{Top}$, with $\star$, Serre fibrations, and weak equivalences;

- chain complexes of sheaves in degree ≥ 0, with injections in degree > 0, surjections with injective kernel, and quasi isomorphisms.
- DGLAs, with $\star$, surjections and quasi isomorphisms.

Lecture 3 (1 hour) September 2nd, 2010

Recall that given a model category $\mathcal{C}$, we defined the homotopy category $\mathrm{H}_0(\mathcal{C})$ formally inverting weak equivalences. We also write $[A, X]$ for $\mathrm{Hom}_{\mathrm{H}_0(\mathcal{C})}(A, X)$.

Note that not all objects in \$ are fibrant, but $\mathrm{Sing}(X)$ is, as is any simplicial abelian group. Moreover, for every $K \in \$$, $K \to \mathrm{Sing}(|K|)$ is a fibrant replacement (that is, is a weak equivalence to a fibrant object).

3.2 *Path objects*

3.4 definition. A *path object* for a fibrant object X in a model category $\mathcal{C}$ is a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & PX \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$

with $X \to PX$ a weak equivalence, and $PX \to X \times X$ a fibration.

Since X is assumed to be fibrant, also PX is fibrant. The axioms of the model category implies the existence of PX, but not its uniqueness.

3.5 example.

1. In $\mathfrak{Top}$, $PX = X^{[0,1]}$, the space of maps $[0, 1] \to X$.
2. In \$, $PX = X^{\Delta^1}$, where, for any object $K \in \$$, $(X^K)_n := \mathrm{Hom}(\Delta^n \times K, X)$.
3. For chain complexes, PV is the cylinder of the map $V \to V \oplus V$ (the underlying graded ring is $V \oplus V \oplus V[-1]$).
4. For DGLAs, $PL = L \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^1|)$.

3.6 theorem. *If A is cofibrant and X is fibrant, then* $[A, X] = {}^{\mathrm{Hom}(A,X)}/_{\mathrm{Hom}(A,PX)}$.

The dual notion of path objects is the notion of *cylinder objects*.

3.3 *Homotopy function objects*

Given $X \in \mathcal{C}$ fibrant, we say that a simplicial diagram $\mathrm{RES}(X)$ in $\mathcal{C}$

$$\begin{tikzcd} X = \mathrm{RES}_0(X) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{RES}_1(X) \arrow[l] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathrm{RES}_2(X) \arrow[l, shift left] \arrow[l, shift right] & \cdots \end{tikzcd}$$

is a *fibrant simplicial resolution* if:

1. $X \to \mathrm{RES}_n(X)$ is a weak equivalence for every n;

2. $\mathrm{RES}_n(X) \to M_n \mathrm{RES}(X)$ is a fibration, where M_n is the n-th Ready matching object.

The boundary $\partial\Delta^n$ of Δ^n is the equalizer of the two maps

$$\prod_{0\le i<j\le n} \Delta^{n-2} \to \coprod_{i=0}^{n} \Delta^{n-1}.$$

We define $M_n \mathrm{RES}(X)$ to be the equalizer of the two maps

$$\prod_{i=0}^{n-1} \mathrm{RES}_{n-1}(X) \to \prod_{0\le i<j\le n} \mathrm{RES}_{n-2}(X).$$

3.7 definition. Given $A, X \in \mathcal{C}$, with X fibrant and A cofibrant, we define the *homotopy function complex* as $\mathrm{RMap}(A, X) \in \$$ by

$$\mathrm{RMap}(A, X)_n := \mathrm{Hom}(A, \mathrm{RES}_n(X)).$$

If X and A are general, we define $\mathrm{RMap}(A, X)$ as $\mathrm{RMap}(A', \hat{X})$, where $\hat{X}$ is a fibrant replacement for X and A' is a cofibrant replacement for A. The result is independent on the choice of the replacementes.

3.8 theorem (Dwyer-Kan). *The construction of* RMap *depends only on* $W \subseteq \mathcal{C}$.

3.9 example.

1. In $\mathfrak{dg}_+\mathfrak{Vect}$, $\mathrm{RES}_n(V) = V \otimes Nk(\Delta^n)$ is generated by non-degenerate i-simplexes in Δ^n in level i.
2. In \$, $\mathrm{RES}_n(X) = X^{\Delta^n}$.
3. For DGLAs, $\mathrm{RES}_n(L) = L \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|)$.

Note that $\mathrm{RES}_1(X)$ is always a path object.

3.4 *MC again*

The following is called *cobar construction*. Given $A \in \mathfrak{dgArt}$, let $\beta^\star(A)$ be the free graded Lie algebra on generators $\mathfrak{m}(A)^\vee[-1]$. The differential is given by $d_A + \Delta\colon \mathfrak{m}(A)^\vee \to \mathfrak{m}(A)^\vee[-1] \oplus \mathfrak{m}(A)^\vee[-2]$, where Δ is the dual of the multiplication.

These are the main properties of the cobar construction:

1. $\beta^\star(A)$ is cofibrant;
2. $\mathrm{MC}(L, A) = \mathrm{Hom}_{\mathfrak{DGla}}(\beta^\star(A), L)$;
3. $\mathrm{Def}(L, A) = [\beta^\star(A), L]$;
4. $\mathrm{MC}(\underline{L}, A) = \mathrm{RMap}(\beta^\star A, L)$.

3.5 *Homotopy fiber products*

3.10 definition. Given $A \to B$ and $C \to B$ with B fibrant, define the *homotopy fiber product* $A \times^h_B C$ as $A' \times_B C'$, where $A \to A'$ is a weak equivalence, $A' \to B$ is a fibration, and similarly for C'.

If $\mathcal{C}$ is right proper (as is every category we have seen), this is equivalent to $A' \times_B C$, that is, we don't need to replace C with C'.

3.11 example. The object $A \times_B PB$ is a replacement for A, fibrant over B. As a corollary, $A \times^h_B C \simeq A \times_B PB \times_B C$.

3.6 *Quillen functors*

3.12 definition. Given two adjoint functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ (that is, we have $\mathrm{Hom}(FA, B) \cong \mathrm{Hom}(A, GB)$), we say that F is *left Quillen* or that G is *right Quillen* if either:

1. F preserve cofibrations and trivial cofibrations, or

2. G preserve fibrations and trivial fibrations.

3.13 definition. Given a right Quillen functor $G\colon \mathcal{D} \to \mathcal{C}$, we define the *right derived functor* $\mathrm{R}\,G$ by $\mathrm{R}\,G(X) := G(X')$, where $X \to X'$ is a fibrant replacement. We define $\mathrm{L}\,F$ for F left Quillen dually.

3.14 example. Let $f\colon X \to Y$ be a map of topological spaces; then we have functors $f_\star\colon \mathfrak{dg}_+\mathfrak{Sh}(X) \to \mathfrak{dg}_+\mathfrak{Sh}(Y)$ and $f^\star$ the other way round. All objects are cofibrant, so $\mathrm{L}\,f^\star = f^\star$; but $\mathrm{R}\,f_\star(V)$ is $f_\star I^\bullet$ for $V \to I^\bullet$ a fibrant replacement, i.e., an injective resolution.

These are some properties of Quillen functors:

1. $\mathrm{RMap}(\mathrm{L}\,FA, B) \simeq \mathrm{RMap}(A, \mathrm{R}\,GB)$;

2. $\mathrm{L}\,F\colon \mathrm{H}_0(\mathcal{C}) \to \mathrm{H}_0(\mathcal{D})$ and $\mathrm{R}\,G\colon \mathrm{H}_0(\mathcal{D}) \to \mathrm{H}_0(\mathcal{C})$ are well defined.

3.15 definition. We say that (F, G) are a *Quillen equivalence* if $\mathrm{L}\,F$ and $\mathrm{R}\,G$ are an equivalence between $\mathrm{H}_0(\mathcal{C})$ and $\mathrm{H}_0(\mathcal{D})$.

If (F, G) are a Quillen equivalence, $\mathrm{RMap}(\mathrm{L}\,FA, \mathrm{L}\,FA') \simeq \mathrm{RMap}(A, A')$ and similarly for $\mathrm{R}\,G$.

3.16 example.

1. The functor $\mathrm{Sing}\colon \mathfrak{Top} \to \$$ and $|\bullet|: \$ \to \mathfrak{Top}$ form a Quillen equivalence.

2. The cotangent complex $\mathbb{L}_{R/B}$ can be describes as the left Quillen functor $\mathfrak{dg}_+\mathfrak{Alg}_B \downarrow_R \to \mathfrak{dgMod}_R$.

4 Derived deformation theory

4.1 *Functor categories*

We observe that $\mathfrak{dgArt}$ is too small to be a model category. The solution is the following.

4.1 definition. We say that $F\colon \mathcal{C} \to \mathcal{D}$ is *left exact* if

$$F(A \times_B C) \cong F(A) \times_{F(B)} F(C)$$

and it preserves the final object. Let $\operatorname{lex}(\mathcal{C}, \mathcal{D})$ be the category of left exact functor $\mathcal{C} \to \mathcal{D}$.

Consider $\operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$; this contains $(\mathfrak{dgArt})^{\mathrm{op}}$ as a full subcategory. To prove this, we can associate to an object $A \in \mathfrak{dgArt}$ the functor $A \mapsto \operatorname{Spec} A$, where $(\operatorname{Spec} A)(B) := \operatorname{Hom}(A, B)$.

4.2 theorem. *There is a model structure on* $\operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ *with these properties:*

1. *all objects are cofibrant;*
2. *F is fibrant (trivially fibrant) if and only if $F(A) \to F(B)$ is surjective for all acyclic semismall extensions (all semismall extensions);*
3. *a map of fibrant objects $F \to G$ is a weak equivalence if and only if the map of univeral deformation functors $F^+ \to G^+$ is an isomorphisms.*

Note that, with this definition of model structure, $\mathrm{MC}(L)$ is fibrant, while $\mathrm{Gg}(L)$ is trivially fibrant.

4.3 lemma. *The object* $\mathrm{MC}(L) \xrightarrow{s} \mathrm{MC}(L) \times \mathrm{Gg}(L) \xrightarrow{t_1, t_2} \mathrm{MC}(L)$ *is a path object.*

Proof. Since $\mathrm{Gg}(L)$ is trivially fibrant, s is a weak equivalence. Then $\mathrm{MC}(L) \times \mathrm{Gg}(L) \to \mathrm{MC}(L) \times \mathrm{MC}(L)$ is a fibration. This says that for every $A \twoheadrightarrow B$ semismall and acyclic, and for every $x, y \in \mathrm{MC}(L, A)$ and $g \in \mathrm{Gg}(L, B)$, such that $g(\overline{x}) = \overline{y} \in \mathrm{MC}(L, B)$, there exists $\widetilde{g} \in \mathrm{Gg}(L, A)$ over g such that $\widetilde{g}(x) = y$. □

4.4 remark. The functors $F \in \operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ are precisely $\mathrm{MC}(V)$, where V is a L_∞-algebra.

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4.5 theorem. *Given a deformation functor $F\colon \mathfrak{dgArt} \to \mathfrak{Sets}$, there exists a fibrant $G \in \operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ such that $F(A) = [\operatorname{Spec} A, G]$ for every $A \in \mathfrak{dgArt}$. This association induces an equivalence between the category of deformation functors and the homotopy category* $\mathrm{H}_0(\operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets}))$.

Outline of the proof. The functor F can be extended to inverse systems setting

$$F(\{A_i\}_{i \in I}) := \varprojlim_{i \in I} F(A_i).$$

By Grothendieck prorepresentability, one proves that objects of $\operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ are $\varinjlim_{i\in I} \operatorname{Spec} A_i$. Hence, F becomes a functor $\operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})^{\mathrm{op}} \to \mathfrak{Sets}$.

Now, we apply Heller's theorem, that states the existence of G such that $[\varinjlim_{i\in I} \operatorname{Spec} A_i, G] = F(\{A_i\}_{i\in I})$ for every $\{A_i\}_{i\in I}$. □

4.6 theorem. *The functor* $\mathrm{MC}\colon \mathfrak{DGla} \to \operatorname{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ *is a right Quillen equivalence.*

Proof. Given $\varinjlim_{i\in I} \operatorname{Spec} A_i$, we get a DGLA $\varinjlim_{i\in I} \beta^\star(A_i)$ using the cobar construction. Then

$$\operatorname{Hom}(\varinjlim_{i\in I}(\beta^\star(A_i)), L) = \varprojlim_{i\in I} \operatorname{Hom}(\beta^\star(A_i), L) = \varprojlim_{i\in I} \mathrm{MC}(L, A_i).$$

So $\beta^\star$ is a left adjoint to MC; but MC preserves fibrations ant trivial fibrations, hence is right Quillen.

The last part of the proof involves spectral sequences. □

Given a DGLA L, let $\beta(A)$ be the inverse system $k[L[1]^\vee]$ (note that $L[i]$ is a pro-finite dimensional vector space). Then, $\mathrm{MC}(L) = \operatorname{Specf} \beta(L)$, and the differential are given on the generators by $d_L + \Delta\colon L[i]^\vee \to L[2]^\vee \oplus \bigwedge^2 L[2]^\vee$, where Δ is the dual to $[\bullet, \bullet]$.

Consider the space $\operatorname{Hom}(\mathrm{MC}(L), \mathrm{MC}(M))$; from what we said, this is equal to $\operatorname{Hom}(\operatorname{Specf} \beta(L), \mathrm{MC}(M))$, hence it is

$$\mathrm{MC}(M, \beta(L)) = \operatorname{Hom}_{\mathfrak{DGla}}(\beta^\star \beta(L), M).$$

Infact, we see that this space is the space of ∞-maps $L \to M$. Therefore, $\operatorname{RMap}(L, M) = \mathrm{MC}(\underline{M}, \beta(L))$. Applying Fiorenza-Martinengo, this gives the Griffiths period map, Bogomolov-Tian-Todorov, the Kodaira embedding principle, Goldman-Millson and other theorems.

Right Quillen functors preserve homotopy fiber products of DGLAs; therefore, given $\chi\colon L \to M$, we have

$$\mathrm{MC}(L \times^h_M \{0\}) = \mathrm{MC}(L) \times^h_{\mathrm{MC}(M)} \{0\}.$$

Also, $\mathrm{MC}(M) \times \mathrm{Gg}(M)$ is a path object for $\mathrm{MC}(M)$, so

$$\mathrm{MC}(L \times^h_M \{0\}) = (\mathrm{MC}(L) \times \mathrm{Gg}(M)) \times_{\mathrm{MC}(M)} \{0\}.$$

Then, Manetti-Fiorenza shows that there is a L_∞-algebra C_χ with $\mathrm{MC}(C_\chi) = (\mathrm{MC}(L) \times \mathrm{Gg}(M)) \times_{\mathrm{MC}(M)} \{0\}$.

In the general framework, this translates to the fact that

$$L \times_M (M \otimes \Omega^\bullet_{\mathrm{dR}}(\Delta^1)) \times_M \{0\}$$

has the same properties (but with quasi isomorphisms).

4.2 *Simplicial functors*

Function complexes in $\mathrm{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$. Take the pullback $A(n)'$ of $k \to \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|)$ via $A \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|) \to \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|)$. In general this is not finite dimensional. So we consider $I := \ker(A(n)' \to A^{\times_K (n+1)})$ and do the following.

4.7 definition. We define $A(n) := \{{}^{A(n)'}/{}_{I^r}\}_r$, and $\mathrm{RES}_n(F)(A) := \varprojlim F({}^{A(n)'}/{}_{I^r})$.

$\mathrm{RES}_n(F)(A)$ is a simplicial fibrant resolution. Write $\underline{F}\colon \mathfrak{dgArt} \to \mathbb{S}$ for the functor defined by $\underline{F}_n := \mathrm{RES}_n(F)$.

4.8 remark. It is not true that $\mathrm{MC}(\underline{L}) = \underline{\mathrm{MC}(L)}$. However, we have a weak equivalence $\mathrm{MC}(\underline{L}) \hookrightarrow \underline{\mathrm{MC}(L)}$ defined by

$$L \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|) \otimes A \to \varprojlim_r {}^{(L \otimes \Omega^\bullet_{\mathrm{dR}}(|\Delta^n|) \otimes A)}\diagup_{I^r}.$$

Moreover,

$$\mathrm{MC}(\underline{L}, A) \simeq \mathrm{RMap}(\beta^\star(A), L) \simeq \mathrm{RMap}(\mathrm{Spec}\, A, \mathrm{MC}(L)) \simeq \underline{\mathrm{MC}(L)}(A).$$

4.9 remark. All objects in $\mathfrak{DGla}$ are fibrant, so $\mathrm{R\,MC} = \mathrm{MC}$; likewise, all objects in $\mathrm{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ are cofibrant, so $\mathrm{L}\,\beta^\star = \beta^\star$.

4.10 theorem. *There is a model structure on* $\mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \mathbb{S})$ *for which all objects are cofibrant and F is fibrant if and only if*

1. *for all semismall extensions $A \twoheadrightarrow B$, $F(A) \to F(B)$ is a fibration;*
2. *for all acyclic semismall extensions $A \twoheadrightarrow B$, $F(A) \to F(B)$ is a trivial fibration.*

A transformation $\eta\colon F \to G$ of fibrant objects is a weak equivalence if and only if $F(A) \to G(A)$ is a weak equivalence for every object $A \in \mathfrak{dg}_+\mathfrak{Art}$.

4.11 theorem. *Let $\mathcal{S}$ be the category of functors $F\colon \mathfrak{dg}_+\mathfrak{Art} \to \mathbb{S}$ such that:*

1. *for every semismall extension $A \twoheadrightarrow B$, and for every $C \to B$, the map*

$$F(A \times_B C) \to F(A) \times^h_{F(B)} F(C)$$

is a weak equivalence;

2. *F preserves weak equivalences;*
3. *$F(k)$ is constructible.*

Then, the map $\mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \mathbb{S}) \to \mathcal{S}$*, given by sending F to a fibrant replacement, is an ∞-equivalence. In particular,* $\mathrm{H}_0(\mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \mathbb{S})) \simeq \mathrm{H}_0(\mathbb{S})$

4.12 remark. An ∞-equivalence is a transformation that induces an equivalence on the homotopy categories plus a condition on how space of maps translates under this transformation. More precisely, we have to impose the condition $\mathrm{RMap}(X, Y) = \mathcal{H}\mathrm{om}(X, Y)$, where $\mathcal{H}\mathrm{om}(X, Y)_n := \mathrm{Hom}(X, Y^{\Delta^n})$.

4.13 theorem. *The functor* $\mathrm{lex}(\mathfrak{dgArt}, \mathfrak{Sets}) \to \mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \$)$ *sending* F *to the restriction* $\underline{F}|_{\mathfrak{dg}_+\mathfrak{Art}}$ *is a right Quillen equivalence.*

Proof. We look at RMap, after constructing a left adjoint. □

So the idea is that $\mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \$)$ is the category of simplicial formal dg-schemes, and corresponds to the category of cosimplicial pro-Artinian dg-rings. Also, $\mathrm{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ corresponds to pro-Artinian dg-rings. The functor $\mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \$) \to \mathrm{lex}(\mathfrak{dgArt}, \mathfrak{Sets})$ is Thom-Whitney.

Summing up, we obtained the following:

$$\begin{tikzcd}
\mathfrak{DGla} \arrow[d, bend right, "\mathrm{MC}"'] & & \mathrm{lex}(\mathfrak{sArt}, \$) \\
\mathrm{lex}(\mathfrak{dgArt}, \mathfrak{Sets}) \arrow[u, bend right, "\beta^\star"'] \arrow[rr, bend right, "F \mapsto \underline{F}"'] & & \mathrm{lex}(\mathfrak{dg}_+\mathfrak{Art}, \$) \arrow[ll, bend right, "\mathrm{Thom}"'] \arrow[u, "N^\star F"'] \\
L_\infty\text{-algebras} \arrow[u, hook] & &
\end{tikzcd}$$

The functor $N \colon \mathfrak{dg}_+\mathfrak{Art} \to \mathfrak{sArt}$ is a right Quillen functor induced by the Dold-Kan normalization, and induces $N^\star$ in the diagram.

Given a dg-manifold X, the associated functor $\mathfrak{dg}_+\mathfrak{Alg} \to \$$ is $\underline{X}$, given by $\underline{X}(A)_n := \mathrm{Hom}(\mathrm{Spec}(A \otimes \Omega^\bullet_{\mathrm{dR}}(\Delta^n)), X)$. This is a 0-truncated geometric stack (as in Toën-Vezzosi) or a derived 0-stack (as in Lurie).