Deformation theory of sheaves
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Contents

1 Deformation of quasi-coherent sheaves

Lecture 1 (1 hour)
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1.1 *The locally free case*

1.1 notation. In the following, we will always consider:

- a surjection of rings $A \twoheadrightarrow A_0$ whose kernel I is square zero (i.e., $I^2 = 0$);
- an A-scheme $X \to \operatorname{Spec} A$;
- a locally free sheaf $\mathcal{V}_0$ of $\mathcal{O}_{X_0}$-modules.

We define $X_0 := X \otimes_A A_0$ and denote with Z the topological space underlying X and X_0 (which is the same for both), so that $X = (Z, \mathcal{O}_X)$ and $X_0 = (Z, \mathcal{O}_{X_0})$ as locally ringed spaces; moreover $\mathcal{O}_X$-modules and $\mathcal{O}_{X_0}$-modules are abelian sheaves over Z.

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The questions we want to address is this: is there a locally free $\mathcal{O}_X$-modules $\mathcal{V}$ such that $\mathcal{V}|_{X_0} \cong \mathcal{V}_0$? If so, how many of them are there?

1.2 example. If $\mathcal{V}_0$ is invertible, then we are infact studying the local geometry of the Picard scheme.

1.3 definition. A *deformation* of $\mathcal{V}_0$ to X is a locally free $\mathcal{O}_X$-module $\mathcal{V}$ together with an isomorphism $\varphi\colon \mathcal{V}|_{X_0} \to \mathcal{V}_0$.

An equivalent definition of a deformation of $\mathcal{V}_0$ is the following: a locally free $\mathcal{O}_X$-module $\mathcal{V}$ together with a map of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{V} \to \mathcal{V}_0$ such that the reduces map $\mathcal{V} \otimes \mathcal{O}_{X_0} \to \mathcal{V}_0$ is an isomorphism.

The main theorem we want to prove about deformations of locally free sheaves is the following.

1.4 theorem. *Given A, A_0, I, X, $\mathcal{V}_0$ as before, let $\mathcal{A} := \mathcal{H}\mathrm{om}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)$. Then:*

1. *there is a class $o(\mathcal{V}_0) \in \mathrm{H}^2(X_0, \mathcal{A})$ such that $o(\mathcal{V}_0) = 0$ if and only if $\mathcal{V}_0$ has a deformation;*

2. *if $o(\mathcal{V}_0) = 0$, the set of isomorphisms classes of deformations is a torsor under $\mathrm{H}^1(X_0, \mathcal{A})$;*

3. *given a deformation $(\mathcal{V}, \varphi)$ of $\mathcal{V}_0$, the set of automorphisms $\alpha\colon \mathcal{V} \to \mathcal{V}$ such that $\alpha|_{\mathcal{V}_0} = \mathrm{id}$, is canonically $\mathrm{H}^0(X_0, \mathcal{A})$.*

Proof. We will prove the theorem in the special case where $X = \operatorname{Spec} B$ is an affine scheme and $\mathcal{V}_0 = \mathcal{O}_{X_0}^{\oplus n}$. In this case, $\mathcal{V}_0$ corresponds to the B_0-modulie $B_0^{\oplus n}$. We need to prove the following.

1. There is a natural morphism $B^{\oplus n} \to B_0^{\oplus n}$ inducing a deformation, and $\mathrm{H}^i(X_0, \mathcal{A}) = 0$ for every $i > 0$.

2. Any two deformations are isomorphic, i.e., for any locally free B-module M, and any isomorphism $\varphi\colon M \otimes_B B_0 \to B_0^{\oplus n}$ there is an isomorphism $B^{\oplus n} \to M$ such that

$$\begin{tikzcd} B^{\oplus n} \arrow[rr] \arrow[dr] & & M \arrow[dl, "\varphi"] \\ & B_0^{\oplus n} & \end{tikzcd}$$

   commutes. In order to prove this, choose arbitrarily elements $e_1, \ldots, e_n \in M$ such that $\varphi(e_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$; by the universal property, $B^{\oplus n} \to M$ sending $(0, \ldots, 0, 1, 0, \ldots, 0)$ to e_i is an isomorphism.

3. Consider $\alpha\colon B^{\oplus n} \to B^{\oplus n}$ such that $\alpha \otimes B_0 = \mathrm{id}$; then:

$$\begin{tikzcd}
0 \arrow[d] & \\
\mathrm{Hom}(B^{\oplus n}, IB \otimes B_0^{\oplus n}) \arrow[d] & \\
\mathrm{Hom}(B^{\oplus n}, B^{\oplus n}) \arrow[d] & A \arrow[l, hook'] \arrow[d] \\
\mathrm{Hom}(B_0^{\oplus n}, B_0^{\oplus n}) & \mathrm{id}, \arrow[l, hook']
\end{tikzcd}$$

where A is the preimage of id; using the same method used in the previous point, we can prove that A consists of automorphisms; also, A is a torsor under

$$\mathrm{Hom}_B(B^{\oplus n}, IB \otimes B^{\oplus n}) = \mathrm{Hom}_{B_0}(B_0^{\oplus n}, IB \otimes B_0^n)$$

; then A can be canonically split by $\mathrm{id} \in A$.

To prove the general case, we start with the last statement. Suppose $(\mathcal{V}, \varphi)$ is a deformation of $\mathcal{V}_0$. Since an endomorphism reducing to the identity is an automorphism, we have the following.

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, I\mathcal{O}_X \otimes \mathcal{V}) \arrow[r, equal] \arrow[d] & \mathcal{A} := \mathcal{H}om_{\mathcal{O}_{X_0}}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0) \\
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{V}) \arrow[d] & \\
\mathcal{H}om_{\mathcal{O}_X}(\mathcal{V}, \mathcal{V} \otimes \mathcal{O}_{X_0}) \arrow[d] & \{\text{red maps}\} \arrow[l, hook'] \arrow[d] \\
\mathcal{H}om_{\mathcal{O}_{X_0}}(\mathcal{V}_0, \mathcal{V}_0) & \mathrm{id} \arrow[l, hook']
\end{tikzcd}$$

where $\mathcal{A}$ is the sheaf of automorphisms of $\mathcal{V}$ restricting to id. The last statement than follows by applying Γ.

The other two statements follow from the affine case, from the third statement we just proved, and from the so called Zariski descent. Alternatively, they can be proven directly with Čech cohomology, using local uniqueness to produce cocycles in $\mathcal{A}$. □

A more high-brow proof is the following. To prove the first statement, we can think of $o(\mathcal{V}_0)$ as the class of the gerbe of deformations of $\mathcal{V}_0$ (over Z). This gerbe is the stack whose objects over $U \subseteq Z$ are the deformations of $\mathcal{V}_0|_U$ and whose isomorphisms are isomorphisms of deformations. It can be shown that this stack has a unique object over any affine $U \subseteq Z$, up to isomorphisms, and the automorphisms sheaf of any object over U is canonically identified with

$\mathcal{A}|_U$. So this is indeed an $\mathcal{A}$-gerbe, and Giraud proved that they are classified by $\mathrm{H}^2(Z, \mathcal{A})$. In other words, $o(\mathcal{V}_0) = 0$ if and only if the gerbe is the trivial gerbe $\mathrm{B}\,\mathcal{A}$.

Also the proof of the second statement can be expressed in this language. Fix a deformation $(\mathcal{V}, \varphi)$, and we use this to study other deformations. If $(\mathcal{V}', \varphi')$ is another deformation, consider the sheaf of isomorphisms of deformations, $\mathcal{I}\mathrm{som}((\mathcal{V}', \varphi'), (\mathcal{V}, \varphi))$. On it, the group $\mathcal{A} := \mathcal{I}\mathrm{som}((\mathcal{V}, \varphi), (\mathcal{V}, \varphi))$ acts on the left, and local uniqueness tells us that this makes it an $\mathcal{A}$-torsor. Now, $\mathcal{A}$-torsor up to isomorphisms are in a bijection with $\mathrm{H}^1(Z, \mathcal{A})$, so there is a bijection between $\mathrm{H}^1(Z, \mathcal{A})$ and the set of deformations of $\mathcal{V}_0$ up to isomorphisms.

Using this language one avoid the choices that are mandatory using Čech cohomology (and avoid to prove that the choices don't change the result); on the other hand, the theory behind is much harder.

1.2 *The general case*

If $\mathcal{V}_0$ is not locally free, we have to impose some flatness conditions that were implicit in the previous case.

1.5 notation. We fix the following notation: $X_0 := X \otimes A_0$, $p\colon X \to \operatorname{Spec} A$, and $\mathcal{F}_0$ for a quasi-coherent $\mathcal{O}_{X_0}$-module that is A_0-flat.

1.6 definition. A *deformation* of $\mathcal{F}_0$ is a quasi-coherent $\mathcal{O}_X$-module that is A-flat, together with an isomorphism $\varphi\colon \mathcal{F}|_{X_0} \to \mathcal{F}_0$.

Even not assuming that A is flat over A_0, we have the following.

1.7 proposition (Local criterion for flatness). *The sheaf $\mathcal{F}$ is A-flat if and only if $\mathcal{F} \otimes \mathcal{O}_{X_0}$ is A_0-flat and the map $p^\star I \otimes \mathcal{F} \to I\mathcal{F}$ is an isomorphism.*

In this case, $p^\star I \otimes \mathcal{F} = I\mathcal{O}_X \otimes \mathcal{F} \cong I\mathcal{F}$.

Theorem 1.4 can be restated in a way that is more prone to generalization. The class $o(\mathcal{V}_0)$ can be thought to be in $\mathrm{Ext}^2_{\mathcal{O}_{X_0}}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)$; the set of deformations can be thought as a torsor under $\mathrm{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)$; the infinitesimal automorphisms can be thought as $\mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)$.

If $\mathcal{F}$ is a deformation of $\mathcal{F}_0$, then reduction and multiplication define a canonical extension

$$0 \to I\mathcal{O}_X \otimes \mathcal{F}_0 \to \mathcal{F} \to \mathcal{F}_0 \to 0$$

in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0)$. We can now state the theorem for non locally free sheaves.

1.8 theorem. *Given A, A_0, I, X, $\mathcal{F}_0$:*

1. *there is a class $o(\mathcal{F}_0) \in \mathrm{Ext}^2_{\mathcal{O}_{X_0}}(X_0, I\mathcal{O}_X \otimes \mathcal{F}_0)$ such that $o(\mathcal{F}_0) = 0$ if and only if $\mathcal{F}_0$ has a deformation;*

2. *if $o(\mathcal{F}_0) = 0$, the set of isomorphisms classes of deformations is a torsor under* $\operatorname{Ext}^1_{\mathcal{O}_{X_0}}(X_0, I\mathcal{O}_X \otimes \mathcal{F}_0)$*;*

3. *given a deformation $(\mathcal{F}, \varphi)$ of $\mathcal{F}_0$, the set of auromorphisms $\alpha\colon \mathcal{F} \to \mathcal{F}$ such that $\alpha|_{\mathcal{F}_0} = \mathrm{id}$, is canonically* $\operatorname{Hom}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0)$.

The proof of the last statement is the same we saw before. For the others, one uses adjunctions to prove that

$$\mathrm{R}\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0) = \mathrm{R}\operatorname{Hom}_{\mathcal{O}_{X_0}}(\mathcal{O}_{X_0} \otimes \mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0).$$

Then one uses the spectral sequence

$$\operatorname{Ext}^q_{\mathcal{O}_{X_0}}(\mathcal{T}\mathrm{or}_p^{\mathcal{O}_X}(\mathcal{O}_{X_0}, \mathcal{F}_0), I\mathcal{O}_X \otimes \mathcal{F}_0) \Rightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0).$$

The sequence $0 \to I\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_{X_0} \to 0$ gives $\mathcal{T}\mathrm{or}_q^{\mathcal{O}_X}(\mathcal{O}_X \otimes \mathcal{F}_0) = I\mathcal{O}_X \otimes \mathcal{F}_0$, so the low degree sequence of the spectral sequence is

$$\begin{aligned} 0 \to \operatorname{Ext}^1_{\mathcal{O}_{X_0}}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0) \to \\ \to \operatorname{Hom}(I\mathcal{O}_X \otimes \mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0) \to \operatorname{Ext}^2_{\mathcal{O}_{X_0}}(\mathcal{F}_0, I\mathcal{O}_X \otimes \mathcal{F}_0) \end{aligned}$$

and the identity in the third space maps to $o(\mathcal{F}_0)$.

2 Sheaves on curves

Lecture 2 (1 hour) August 31st, 2010

Our next goal is to prove that the moduli spaces of locally free sheaves with fixed determinant and rank on a curve are unirational.

We will do this without working with geometric invariant theory and without using too much the actual moduli spaces.

2.1 NOTATION. We will write $p\colon C \to S$ for a proper, smooth morphism of schemes, of relative dimension 1 (i.e., a family of smooth curves) with S connected and with connected geometric fibers.

2.2 DEFINITION. We define $\mathfrak{Sh}_{C/S}$ as the stack of locally free sheaves. Objects over $T \to S$ are locally free $\mathcal{O}_{C_T}$-modules (on $C \times_S T$). We define also the closed and open substack $\mathfrak{Sh}^n_{C/S}$ of rank n sheaves.

2.3 EXAMPLE. We have $\mathfrak{Sh}^1_{C/S} = \mathrm{Pic}_{C/S}$, the stack of invertible sheaves.

Technically, these are Artin stacks, and $\mathfrak{Sh}^n_{C/S}$ is locally of finite presentation. This is the hidden reason to explain why the deformation theory is nice.

2.4 THEOREM. *For every n, the stack $\mathfrak{Sh}^n_{C/S}$ is smooth.*

2.5 THEOREM. *Given n, the determinant morphism $\mathfrak{Sh}^n_{C/S} \to \mathrm{Pic}_{C/S}$ is smooth with unirational geometric fibers.*

Let us clarify some notions about the determinant morphism and deformations with fixed determinant. If $\mathcal{V}$ is a locally free sheaf on a scheme Y of rank n, there is a natural invertible sheaf associated, $\det \mathcal{V} := \bigwedge^n \mathcal{V}$; this is a functorial construction, so it defines the morphism we saw in the theorem.

2.1 *Deformation theory with fixed determinant*

Previously we saw the deformation theory for locally free sheaves; now we want to fix also the determinant, so we have $\mathcal{V}_0$ locally free on X_0, and in addition to what we did, we fix also $\mathcal{L}$ invertible on X and an isomorphism $\alpha\colon \det \mathcal{V}_0 \to \mathcal{L}|_{X_0}$.

2.6 definition. A *deformation* of $\mathcal{V}_0$ with determinant $\mathcal{L}$ is a triple (V, φ, γ) where $(\mathcal{V}, \varphi)$ is a deformation of $\mathcal{V}_0$ and $\gamma\colon \det \mathcal{V} \to \mathcal{L}$ is an isomorphism such that $\gamma|_{X_0} = \alpha \cdot \det \varphi$

Even if this is a different deformation problem, we can express the deformation theory in a very similar way. Before showing it, note that the trace defines a map

$$\mathrm{Tr}\colon \mathcal{H}\mathrm{om}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0) \to I\mathcal{O}_X.$$

Moreover, we have the following.

2.7 lemma. *Giving an infinitesimal isomorphism $\alpha\colon \mathcal{O}_X \to I\mathcal{O}_X \otimes \mathcal{O}_{X_0}^{\oplus n}$, we have that* $\det(\mathrm{id} + \alpha) = \mathrm{id} + \mathrm{Tr}(\alpha)$ *as endomorphisms of $\mathcal{O}_X$.*

So the infinitesimal isomorphisms that does not change the determinant are precisely the one with no trace.

2.8 definition. Given i, write $\mathrm{Ext}^i(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)_0$ for the kernel of the trace map

$$\mathrm{Tr}\colon \mathrm{Ext}^i(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0) \to \mathrm{H}^i(X_0, I\mathcal{O}_X);$$

this is called the *traceless part*.

2.9 theorem. *Given $\mathcal{V}_0$, $\mathcal{L}$, and an isomorphism $\alpha\colon \det \mathcal{V}_0 \to \mathcal{L}|_{X_0}$, then:*

1. *the obstruction to deforming with determinant $\mathcal{L}$ lies in* $\mathrm{Ext}^2(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)_0$*;*
2. *isomorphism classes of deformations are a (pseudo) torsor under* $\mathrm{Ext}^1(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)_0$*;*
3. *infinitesimal automorphisms of $(\mathcal{V}, \varphi, \gamma)$ are* $\mathrm{Hom}(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0)_0$

Proof of theorem 2.4. Smoothness means that given a square zero extension $A \twoheadrightarrow A_0$ of affine schemes over S, and given $\mathcal{V}_0$ on C_{A_0}, there exists $\mathcal{V}$ on C_A and an isomorphism $\mathcal{V}|_{C_{A_0}} \to \mathcal{V}_0$. We can do this because we know the obstruction to doing so lies in $\mathrm{Ext}^2(\mathcal{V}_0, I\mathcal{O}_X \otimes \mathcal{V}_0) = \mathrm{H}^2(C_{A_0}, \mathcal{A})$, and proving the following lemma. □

2.10 lemma. *The group* $\mathrm{H}^2(C_{A_0}, \mathcal{F})$ *vanishes for any coherent sheaf* $\mathcal{F}$ *on* C_{A_0}.

Proof. Reduce to the case there A is local and Noetherian; then reduce to A complete by the theorem on flatness of completion. Then the theorem on formal functions tells us that

$$\mathrm{H}^2(C_{\hat{A}}, \mathcal{F}) = \varprojlim \mathrm{H}^2(C_{A/\mathfrak{m}_A^{n-1}}, \mathcal{F}|_{C_{A/\mathfrak{m}_A^{n-1}}}).$$

Then we can do an induction on length to reduce to $A = K$ a field. □

To prove the smoothness of the determinant map one has just to observe that the obstructions are the traceless obstructions, but since there are no obstructions there are also no traceless ones.

2.2 *Unirationality of geometric fibers of the determinant*

To prove unirationality, one has to find a family that contains almost all objects and is parametrized by a rational variety.

More precisely, we will work over $S = \operatorname{Spec} K = \operatorname{Spec} \overline{K}$, with a fixed invertible sheaf $\mathcal{L}$ over C. Consider $\mathfrak{Sh}^n_{C/S}(\mathcal{L})$, the moduli space of pairs $(\mathcal{V}, \gamma\colon \det \mathcal{V} \xrightarrow{\sim} \mathcal{L})$; we want to prove that this is a union of quasi compact open substacks $\mathfrak{Sh}^n_{C/S}(\mathcal{L})_m$, each of which is the image of an open subset of an affine space.

We will prove this using the following Bertini-like theorem.

2.11 proposition. *Suppose* $\mathcal{V}$ *is a locally free sheaf of rank* n *and determinant* $\mathcal{L}$. *Then, for large enough* m, *a general map* $\mathcal{O}(-m)^{n-1} \to \mathcal{V}$ *has cokernel equal to* $\mathcal{L}(m(n-1))$.

Proof. Let m be large enough so that $\mathcal{V}(m)$ is generated by global sections, and $\mathrm{H}^1(C, \mathcal{V}(m)) = 0$. In this case, the functor on K-schemes, sending T to $\mathrm{Hom}_{C_T}(\mathcal{O}(-m)^{n-1}, \mathcal{V})$ is represented by a vector bundle $V \to \operatorname{Spec} K$.

On $C \times V$ there is a universal map $\Phi\colon \mathcal{O}_{C_V}(-m)^{n-1} \to \mathcal{V}_{C_V}$. Fix some $c \in C$; since $\mathrm{H}^0(\mathcal{V}(m))$ generates $\mathcal{V}(m)$, there is a surjection

$$V_c \twoheadrightarrow \mathrm{Hom}(\mathcal{O}(-m)^{n-1}_c \otimes k(c), \mathcal{V} \otimes k(c)) \cong M_{n\times(n-1)}(k(c)).$$

In this space of matrices, the locus of maps with non maximal rank has codimension at least 2 (for $n \geq 2$). Now, on $C \times V$, we have that the locus of points (c, v) such that $\Phi_{(c,v)}$ has non maximal rank has codimension at least 2. But C has dimension 1, so projecting to V, the image of the locus with non maximal rank has codimension at least 1, hence there is an open subset of V such that the map has maximal rank for every point of C. □

As a consequence, we can dominate $\mathfrak{Sh}^n_{C/S}(\mathcal{L})$ by an open subset of the affine space underlying the vector space $\mathrm{Ext}^1_C(\mathcal{L}(m(n-1)), \mathcal{O}(-m)^{n-1})$.

End of proof of Theorem 2.5. For every m, let $\mathfrak{Sh}^n_{C/S}(\mathcal{L})_m$ be the open substack parametrizing $\mathcal{V}$ such that $\mathrm{H}^1(C, \mathcal{V}(m)) = 0$ and $\mathrm{H}^0(C, \mathcal{V}(m))$ generates $\mathcal{V}(m)$. In other words, such that $p^\star p_\star \mathcal{V}(m) \to \mathcal{V}(m)$ is surjective; thanks to this description we know that this locus is open by cohomology and base change.

Let W be the affine space underlying $\mathrm{Ext}^1_C(\mathcal{O}(-m)^{n-1}, \mathcal{L}(m(n-1)))$. Because cohomology commutes with flat base change, there is a universal extension

$$0 \to \mathcal{O}(-m)^{n-1}_{C_W} \to \mathcal{V}_{\mathrm{univ}} \to \mathcal{L}(m(n-1))|_{C_W} \to 0$$

and this implies that $\det \mathcal{V}_{\mathrm{univ}} \cong \mathcal{L}$.

This gives a family, hence a morphism $W \to \mathfrak{Sh}^n_{C/S}(\mathcal{L})_m$, and this morphism is surjective. □

2.12 corollary. *Let $\mathcal{V}$ be a vector bundle over C; then a general deformation of $\mathcal{V}$ is a stable vector bundle.*

Proof. Stability is an open and nonempty condition. Indeed this is the only thing we will use about stability (and the proof works for every open property of vector bundles). Since $\mathfrak{Sh}^n_{C/S}(\mathcal{L})$ is irreducible, then stability is dense. □

3 Sheaves on surfaces

Lecture 3 (1 hour) September 1st, 2010

3.1 *Good sheaves*

3.1 notation. In the following, $f\colon X \to \operatorname{Spec} k$ will be a smooth projective surfaces over a field, that one can imagine to be algebraically closed, but it is not necessary.

The goal is to prove the irreducibility of the moduli spaces of locally free sheaves on X. This problem is more complicated than the previous one also because we have one more Chern class to account for: the invariants will be the rank, the determinant, and the second Chern class c_2.

An optimistic question would be: is the stack $\mathfrak{Sh}^n_{X/k}(\mathcal{L}, c)$ of locally free sheaves on X with determinant $\mathcal{L}$ and second Chern class c irreducible, unirational, of general type, or smooth?

The answer is less optimistic: most of the time it is "no", even if sometimes it is "yes".

From the history of moduli space, one learns that even if one is interested only in the moduli of some nice problem, he has to compactify the space with degenerate objects that maybe are not even interesting in themselves. In particular, there is a sequence of nice problems that can be packaged inductively by degeneration.

3.2 example. Consider M_g, the stack of smooth curves of genus g; Deligne and Mumford proved that this is irreducible, but it is not compact. One can construct a compactification, $\overline{M}_g$, where the boundary is a union of pieces made from M_h, with $h < g$. Using the compactification, we can prove the irreducibility in another way: since M_g is smooth, we only need to prove it is connected; so for each pair of points, we can find two curves to connect the each point with the boundary, and use induction to connect the two endpoints in the boundary.

In our case, we won't really construct a compactification, but we will add the points corresponding to certain torsion free sheaves that are not locally free. Note that the terminology in the following is sightly different from the one used in the literature.

3.3 definition. A torsion free sheaf $\mathcal{F}$ on X is *good* if $\mathrm{Ext}^2(\mathcal{F}^{\star\star}, \mathcal{F}^{\star\star})_0 = 0$.

The sheaf $\mathcal{F}^{\star\star}$ is the double dual of $\mathcal{F}$ and it is always locally free; the subscript means that we want only the traceless part to be 0.

When $\mathcal{F}$ is already locally free, a good sheaf is a sheaf for which the obstruction group is trivial. But when $\mathcal{F}$ is just torsion free, this again says that the obstruction group is trivial, but also says *more*.

Before going on, for simplicity we will make another assumption, that the rank of our sheaves are units in the field k; i.e., we are not dealing with rank p sheaves in characteristic p.

Let us specify what was that "more": it is that $\mathrm{H}^2(X, \mathcal{E}\mathrm{nd}(\mathcal{F})_0) = 0$. As we will see, this is the obstruction space to a local to global problem that we will have to deal with in the following.

3.4 proposition. *If $\mathcal{F}$ is good and $\mathrm{rk}\,\mathcal{F} \geq 2$, then there is a locally free deformation of $\mathcal{F}$ over $k[[t]]$.*

Sketch of the proof.

1. Make an affine open covering $\{U_i\}$ of X and consider sheaves $\mathcal{F}_i$ on $U_i \otimes k[[t]]$ such that $\mathcal{F}_i|_{t=0} \cong \mathcal{F}|_{U_i}$, and $\mathcal{F}_i|_{k((t))}$ is locally free.
2. There are associated coherent sheaves $\hat{\mathcal{F}}_i$ on the formal schemes $\widehat{U_i \otimes k[[t]]}$.
3. The obstruction to glueing the sheaves $\hat{\mathcal{F}}_i$, one thickening at a time, lies in $\mathrm{H}^2(X, \mathcal{E}\mathrm{nd}(\mathcal{F})_0)$, which is zero by assumption.
4. Therefore, there exists a coherent sheaf on $\widehat{X \otimes k[[t]]}$.
5. Grothendieck existence (EGA III) algebraizes this construction. □

Grothendieck existence, or formal GAGA, states that if Y/A is proper or complete and locally Noetherian, then $\mathfrak{Coh}(Y) = \mathfrak{Coh}(\hat{Y})$ is an equivalence of categories. Concretely, given a sequence of sheaves $\mathcal{F}_i$ on $Y \otimes A/\mathfrak{m}^{i+1}$, and isomorphisms $\mathcal{F}_i|_{Y \otimes A/\mathfrak{m}^i} \to \mathcal{F}_{i-1}$, then there is a unique $\mathcal{F}$ on Y with compatible isomorphisms $\mathcal{F}|_{Y \otimes A/\mathfrak{m}^{i+1}} \to \mathcal{F}_i$

We could prove that being good is an open condition, i.e., constructible and stable under generization. Therefore, the moduli problem of good sheaves is good.

3.2 *Inductive structure*

The goal now is to make new good sheaves from old ones.

Take $\mathcal{F}$, and choose a point $x \in X$ over which $\mathcal{F}$ is locally free. Take a quotient $\mathcal{F} \otimes k(x) \to k(x)$ and let $\mathcal{F}' := \ker(\mathcal{F} \to \mathcal{F} \otimes k(x) \to k(x))$.

The new sheaf is no more locally free over x, but it has the same reflexive hull (double dual) of $\mathcal{F}$. Moreover, we could prove with a computation that $c_2(\mathcal{F}') = c_2(\mathcal{F}) + 1$, and also the determinant does not change, being the determinant of $\mathcal{F}$ isomorphic to the determinant of $\mathcal{F}^{\star\star}$.

So with this construction we are relating different moduli spaces of good sheaves; in particular, we have a map $\mathfrak{Good}^n_X(\mathcal{L}, c) \to \mathfrak{Good}^n_X(\mathcal{L}, c+1)$, that is analogous to the natural maps we have between the moduli spaces of curves.

Write $\Xi(c)$ for the set of the components of $\mathfrak{Good}^n_X(\mathcal{L}, c)$; then the previous morphism gives a map $\Xi(c) \to \Xi(c+1)$. Let Φ be the degree 1 map that account for all these maps: $\Phi\colon \coprod_c \Xi(c) \to \coprod_c \Xi(c)$.

3.5 theorem (O'Grady). *Given a finite subset $\xi \subseteq \Xi(c)$, there exists d such that $\Phi^{\circ d}(\xi)$ is a singleton; i.e., Φ is a contraction.*

3.6 theorem (O'Grady). *For the substack of stable sheaves, we have:*

1. *for $c \gg 0$, Φ is surjective on components;*

2. *for $c \gg 0$, $\mathfrak{Stable}^n_X(\mathcal{L}, c)$ is (geometrically) irreducible.*

The key point to prove the two theorems is another Bertini-like theorem.

3.7 theorem. *Suppose V, W are two locally free sheaves on X of rank n, determinant $\mathcal{L}$ and $c_2 = c$. For large enough m, the cokernel of a general map $V \to W(m)$ is an invertible sheaf Q supported on a smooth member of $|\mathcal{O}(nm)|$. Moreover, the degree of Q depends only on n, $\mathcal{L}$, c.*

In other words, we have $0 \to V \to W(m) \to Q \to 0$, where $Q \in \operatorname{Pic}(C)$ and $C \subseteq X$ is a smooth divisor in $|\mathcal{O}(nm)|$; moreover, $\deg Q = d$ is independent of V and W.

As we vary C and Q, the spaces $\operatorname{Ext}^1_X(Q, V)$ do not form a vector bundle; but for $l \gg 0$, the spaces $\operatorname{Ext}^1_X(Q(-l), V)$ do. We end up with the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V \arrow[r] \arrow[d, equal] & W' \arrow[r] \arrow[d, hook] & Q(-l) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & V \arrow[r] \arrow[d] & W(m) \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & S \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

where S has finite length d. Hence, $[W'] \in \mathfrak{Good}^n_{X/k}(\mathcal{L}, c+l)$.

4 An application to the geometry of $\mathbb{P}^n$-bundles

Lecture 4 (1 hour) September 2nd, 2010

4.1 *Measuring the failure to be a $\mathbb{P}^n$-bundle*

Before going on, let us summarize what we have seen so far.

1. We described the infinitesimal deformation theory of sheaves (and the generality of the discussion hints to the fact that the theory applies also to more general spaces than curves and surfaces).
2. We showed that if Y is a curve over a field, then the spaces $\mathfrak{Sh}^n_Y(\mathcal{L})$ are geometrically irreducible and geometrically unirational.
3. We showed that if Y is a surface, then the spaces $\mathfrak{Sh}^n_Y(\mathcal{L}, c)$ contain geometrically irreducible open substacks.

Now we are going to discuss $\mathbb{P}^n$-bundles, that is, proper flat morphisms $P \to X$ with geometrical fibers isomorphic to $\mathbb{P}^n$.

4.1 example. If $\mathcal{V}$ is a locally free sheaf of rank $n+1$, then $P = \mathbb{P}(\mathcal{V})$ is a $\mathbb{P}^n$-bundle. But not all $\mathbb{P}^n$-bundle arise in this way: let $X := \operatorname{Spec} \mathbb{R}$, $P := Z(x^2 + y^2 + z^2) \subseteq \mathbb{P}^2$; P is a conic without rational points, so it is different from $\mathbb{P}^1$.

4.2 problem. Given $P \to X$, how can we measure the failure of P to be (Zariski locally on X) of the form $\mathbb{P}(\mathcal{V})$?

We can rephrase the question in this way: let $X = \operatorname{Spec} K$, P be defined over K and $P \otimes \overline{K} \cong \mathbb{P}^n$; how can we tell how far P is from $\mathbb{P}^n$?

4.3 remark. There are two ways to tell if $P \cong \mathbb{P}^n$:

1. if there exists a divisor $H \subseteq P$ of degree 1, or in other words if $H \otimes \overline{K}$ is a hyperplane, then $|\mathcal{O}(H)| : P \to \mathbb{P}^n$ is an isomorphism;
2. if $P(K) \neq \varnothing$, then $P \cong \mathbb{P}^n$; more refined, if P contains a 0-cycle of degree 1 over K, then $P \cong \mathbb{P}^n$.

The reason for the second way to be true, is that every P has a dual fibration, $P^\vee$; under this duality, points becomes hyperplanes and viceversa; therefore, $P(K) \neq \varnothing$ implies that there exists $H \in P^\vee$ such that $H \otimes \overline{K}$ is an hyperplane, hence $P^\vee \cong \mathbb{P}^n$ and consequently $P \cong \mathbb{P}^n$.

For the second part, a 0-cycle of degree 1 implies the existence of a sequence of field extension ${}^{L_i}/_{K}$ with $\gcd\{[L_i : K]\} = 1$ and the existence of hyperplanes $H_i \subseteq P^\vee \otimes L_i$ for each i. The norm $N_{L_i/K}(H_i) \subseteq P$ has degree $[L_i : K]$, hence an appropriate linear combination $\sum a_i N_{L_i/K}(H_i) \subseteq P$ is a divisor in $\mathcal{O}(1)$ (over $\overline{K}$). This means that $\mathcal{O}(1)$ is rational over K, hence $\overline{\mathcal{O}(1)} \colon P \xrightarrow{\sim} \mathbb{P}^n$.

So we have two ways to measure the distance from P to $\mathbb{P}^n$:

1. we may consider d, the minimum positive integer such that there exists $D \subseteq P$ with $D \otimes \overline{K} \subseteq |\mathcal{O}(d)|$; d is called the *period* of D and denoted by $\operatorname{per}(P)$;

2. we can consider i, the minimum positive degree of a 0-cycle on P; i is called the *index* of P and denoted by $\mathrm{ind}(P)$.

4.2 *Relation between index and period*

Note that there is no easy relation between the two (for example, the index is not the period of the dual). Indeed, the index measure the existence of hyperplanes after field extensions, while the period measure the existence of divisor of low degree.

But the norm argument shows that $\mathrm{per}(P) \mid \mathrm{ind}(P)$; also, (only when K is infinite) if $D \subseteq P$ is geometrically $\mathcal{O}(d)$, then $D^n = d^n$, hence general members of $|D|$ will interact to give a 0-cycle of degree d^k; this gives $\mathrm{ind}(P) \mid \mathrm{per}(P)^n$.

One point of view we can pursue now is to find classes of situations where $\mathrm{per}(P) = \mathrm{ind}(P)$; for example, we may ask if there exists a field K where equality holds, or maybe a dimension.

4.4 example. If $n = 1$, then $\mathrm{per}(P) = \mathrm{ind}(P)$. Infact, this number is either 1 or 2, depending on the presence of rational points in the conic P. The reason here is that divisors and 0-cycles coincides on curves.

4.5 example. If $K = \overline{K}$, then $\mathrm{per}(P) = \mathrm{ind}(P) = 1$, that is, we always have $P \cong \mathbb{P}^n$. This is also true if $K = \mathbb{F}_q$ (Wedderburn). Also if $K = k(C)$ is the function field of a curve C/k with $k = \overline{k}$, then $P \cong \mathbb{P}^n$ (Tsen).

4.6 example. There are cases where $\mathrm{ind}(P) \mid \mathrm{per}(P)^n$ is sharp; for example, consider a twisted Segre map $P_1 \times \cdots \times P_l \hookrightarrow P$; if $K = k(a_1, a_2, \ldots, a_{2n-1}, a_{2n})$, let $P_i = Z(x^2 - a_{2i-1}y^2 - a_{2i}z^2)$. Then $\mathrm{per}(P) = 2$, while $\mathrm{ind}(P) = 2^l$. This is not obvious, but shows that when the fields are generic, also the results are.

4.7 theorem (de Jong). *If K is the function field of a surface over $k = \overline{k}$, then* $\mathrm{per}(P) = \mathrm{ind}(P)$ *for every P.*

4.8 theorem (Lieblich). *If $P \to X$ is a projective bundle over a proper surface over $\mathbb{F}_q$, then* $\mathrm{per}(P) = \mathrm{ind}(P)$. *This is not true if X is not proper, but in this case* $\mathrm{ind}(P) \mid \mathrm{per}(P)^2$.

The second part is significantly harder; the first relies on the reduction properties over a compact surface.

We have said that a $\mathbb{P}^n$-bundle $P \to X$ is not necessarily the projectivization of a locally free sheaf. But we can introduce another object where this is true: there exists a stack $\mathfrak{X}$ over X, and a locally free sheaf $\mathcal{V}$ over $\mathfrak{X}$ such that $P_{\mathfrak{X}} \cong \mathbb{P}_{\mathfrak{X}}(\mathcal{V})$; in other words, we have a cartesian diagram

$$\begin{tikzcd} P_{\mathfrak{X}} \cong \mathbb{P}_{\mathfrak{X}}(\mathcal{V}) \arrow[r] \arrow[d] & P \arrow[d] \\ \mathfrak{X} \arrow[r] & X. \end{tikzcd}$$

We can interpret $\mathfrak{X}$ as a moduli space: it corresponds to the moduli problem of pairs $(\mathcal{V}, \varphi)$ with $\varphi\colon P \xrightarrow{\sim} \mathbb{P}(\mathcal{V})$. This stack is a $\mathbb{G}_m$-gerbe (we can think of it as a $B\mathbb{G}_m$-bundle over X). For us, this is like X but with a $\mathbb{G}_m$ group of automorphisms at each point and a nontrivial twist over them.

Vector bundles on $B\mathbb{G}_m$ corresponds to representations of $\mathbb{G}_m$; hence $\mathbb{G}_m$ acts on the fibers of $\mathcal{V}$. Here, since they have to be compatible with the identification φ, the automorphisms of a pair are just scalar multiplications.

4.9 PROPOSITION. *The index of P divides n if and only if there is a locally free $\mathcal{V}$ on $\mathfrak{X}$ of rank n such that $\mathbb{G}_m$ is* twisted *(i.e., acts by scalar multiplication).*

Denote with $\mathfrak{Sh}^n_{\mathcal{Y}}(\mathrm{tw})$ the moduli space of twisted locally free sheaves of rank n over $\mathcal{Y}$. If $\mathrm{per}(P) \mid n$, then to show $\mathrm{ind}(P) = n$ we need only to prove that $\mathfrak{Sh}^n_{\mathcal{Y}}(\mathrm{tw})(K) \neq \varnothing$. We have translated the problem into a moduli question. We are going to use this formulation to prove the two theorems. For simplicity, we always assume what $P \to X$ is proper.

Proof of de Jong's theorem. Blow up one fiber X; then we obtain $\widetilde{P} \to \widetilde{X}$ and $\widetilde{X} \to \mathbb{P}^1$ is a family of projective bundles over curves parametrized by $\mathbb{P}^n$:

$$\begin{tikzcd} \widetilde{P} \arrow[r] & \widetilde{X} \arrow[d, shift left] \\ & \mathbb{P}^1 \arrow[u, hook, shift left] \end{tikzcd}$$

We are interested in the generic fiber

$$\begin{tikzcd} P_\eta \arrow[r] & X_\eta \arrow[d] \\ & \operatorname{Spec} k(t) \end{tikzcd}$$

The first thing to do is to base change to $\overline{k(t)}$; then we have a projective bundle over an algebraically closed field, $\overline{P}_\eta \to \overline{X}_\eta$. Tsen showed that in this case $\overline{P}_\eta$ is trivial, therefore $\mathrm{ind}(\overline{P}_\eta) \mid n$ and so $\mathfrak{Sh}^n_{\mathfrak{X}_\eta}(\mathrm{tw})(\overline{k(t)}) \neq \varnothing$.

But now we know that $\mathfrak{Sh}^n_{\mathfrak{X}_\eta} \otimes \overline{k(t)}$ is unirational and that our moduli problem is geometrically rationally connected over $k(t)$. Hence, by Graber-Harris-Stan, it has a point, therefore $\mathrm{ind}(P) = \mathrm{ind}(P_\eta) \mid \mathrm{per}(P)$. $\square$

Proof of Lieblich's theorem. Consider $\mathfrak{Sh}^n_{\mathfrak{X}}(\mathrm{tw})$ over $\mathbb{F}_q$. By de Jong's theorem, we know that over $\overline{\mathbb{F}}_q$ it is nonempty. We have shown that $\mathfrak{Sh}^n_{\mathfrak{X}}(\mathrm{tw})$ contains a geometrically closed substack S. Over a finite field, the Lang-Weil estimate implies that S has a 0-cycle of degree 1, therefore $\mathrm{ind}(P) \mid \mathrm{per}(P)$. $\square$