

ELEMENTARY DEFORMATION THEORY

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1 INTRODUCTION

This will be an introductory course. For this reason, every scheme will be of finite type over an algebraically closed field K .

*Lecture 1 (1 hour)
August 30th, 2010*

Let us review some basic ideas in modern algebraic geometry:

1. studying morphisms, not objects;
2. studying a scheme X by its functor of points, i.e. by the collection of morphisms $\text{Spec } A \rightarrow X$.

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1.1 Fat points

1.1 EXAMPLE. Assume that $f: X \rightarrow Y$ is a morphism of scheme; then f is proper if and only if for every commutative diagram

$$\begin{array}{ccc} C_0 := C \setminus \{p\} & \longrightarrow & X \\ \downarrow & & \downarrow f \\ C & \longrightarrow & Y \end{array}$$

with C a smooth (affine) curve and $p \in C$ a closed point, i.e. $p: \text{Spec } K \rightarrow C$, we have a unique morphism $C \rightarrow X$ commuting with the diagram. This is called the *valuative criterion for properness*.

1.2 EXERCISE. Check how this criterion relates to other versions of the valuative criterion.

1.3 COROLLARY. *Properness and separatedness do not see the scheme structure. In other words, $f: X \rightarrow Y$ is proper if and only if $f_{\text{red}}: X_{\text{red}} \rightarrow Y_{\text{red}}$ is proper.*

On the other end, the most basic object that has a scheme structure is the following.

1.4 DEFINITION. A *fat point* is a scheme S such that $S_{\text{red}} \cong \text{Spec } K$.

A fat point has the simplest topology (it is just a point), therefore is the simplest object with a nontrivial scheme structure.

1.5 EXAMPLE. Let $D := \text{Spec } K[\varepsilon]/\varepsilon^2$; let X be a scheme and $x \in X$ a point. There is a natural bijection between $T_x X$ and the set of morphisms $D \rightarrow X$ such that

$$\begin{array}{ccc} D & \longrightarrow & X \\ \uparrow & & \uparrow x \\ D_{\text{red}} & \xrightarrow{\sim} & \text{Spec } K \end{array}$$

commutes. We can indeed identify such a morphism with a tangent direction at x .

1.6 REMARK. If S is a fat point, then S is affine (so $S = \text{Spec } A$); conversely, $\text{Spec } A$ is a fat point if the following conditions are satisfied (note that this is far from being a minimal set of conditions):

1. it has to be a local K -algebra (so it has a unique maximal ideal \mathfrak{m}_A);
2. $A/\mathfrak{m}_A \cong K$ as a K -algebra;
3. \mathfrak{m}_A is nilpotent (i.e., $A_{\text{red}} \cong K$); by Nakayama's Lemma, this implies that A is finite dimensional over K as a vector space (the point is that

each quotient $\mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}$ is finite dimensional and finitely many of them are nonzero);

4. being a finite dimensional K -algebra, A is Artinian, i.e. the descending chain condition holds for ideals in A .

1.7 DEFINITION. Let \mathfrak{Art} be the category whose objects are K -algebras satisfying the previously stated conditions, and whose morphisms are local K -algebra homomorphism.

In particular, note that the assumptions we have on the objects imply that the homomorphisms are local.

To every scheme X , we can associate its functor of points $h_X: K\text{-alg} \rightarrow \mathfrak{Sets}$. We define $h_X(A) := \text{Mor}_{K\text{-sch}}(\text{Spec } A, X)$, and, for any morphism of K -algebras $\pi: \tilde{A} \rightarrow A$, the map of set $h_X(\pi)$ sends $f: \text{Spec } \tilde{A} \rightarrow X$ to the composition $\text{Spec } A \rightarrow \text{Spec } \tilde{A} \rightarrow X$.

The question is: how much of $X \rightarrow Y$ can we know if we restrict h_X, h_Y , and $h_X \rightarrow h_Y$ to \mathfrak{Art} ?

1.8 PROPOSITION. *The morphism $F: X \rightarrow Y$ is smooth (étale) if and only if for every commutative diagram*

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & X \\ \downarrow i & & \downarrow \\ \text{Spec } \tilde{A} & \longrightarrow & Y \end{array}$$

with $A, \tilde{A} \in \mathfrak{Art}$, and $\tilde{A} \rightarrow A$ surjective (i.e., i is a closed embedding), there exists (exists and is unique) a morphism $\tilde{f}: \text{Spec } \tilde{A} \rightarrow X$ commuting with the diagram.

We will prove this proposition later; one usually refers to it as the fact that smoothness (étaleness) is equivalent to formal smoothness (étaleness).

The aim of these lectures is to study this kind of questions, but moreover we will apply these methods to other context, namely the infinitesimal study of morphisms of moduli spaces and moduli stacks.

1.2 Extensions

1.9 DEFINITION. Let $\pi: \tilde{A} \rightarrow A$ be a surjective morphism in \mathfrak{Art} ; if we define $I := \ker \pi$, we have an exact sequence

$$(1) \quad 0 \rightarrow I \rightarrow \tilde{A} \xrightarrow{\pi} A \rightarrow 0.$$

We say that (1), or π , is a *square zero extension* of A by I if $I^2 = 0$.

1.10 REMARK. If $\pi: \tilde{A} \rightarrow A$ is a square zero extension, then I is naturally an A -module. The real reason to define these particular kind of extensions is

that one can work out deformation theory working only with these. This has actually been done in the original paper by Illusie.

1.11 DEFINITION. An extension (1) or a surjection $\pi: \tilde{A} \rightarrow A$ is called *semismall* if $I \cdot \mathfrak{m}_{\tilde{A}} = 0$, i.e. if I , as an \tilde{A} -module, is just a K -vector space.

Note that $I \cdot \mathfrak{m}_{\tilde{A}} = 0$ actually implies $I^2 = 0$.

1.12 DEFINITION. A semismall extension is called *small* if $\dim_K I = 1$.

1.13 EXAMPLE. Let

$$\pi: K[t]_{/t^{N+1}} \rightarrow K[t]_{/t^{M+1}},$$

with $N > M$. Then π is square zero if and only if $t^{M+1} \cdot t^{M+1} = 0$, i.e. if $2M + 2 \geq N + 1$; it is small if and only if it is semismall if and only if $N = M + 1$.

1.14 REMARK. For every $N > M$, π factors as a composition of small extensions:

$$K[t]_{/t^{N+1}} \rightarrow K[t]_{/t^N} \rightarrow \dots \rightarrow K[t]_{/t^{M+1}}.$$

This property is actually more general.

1.15 PROPOSITION. Every surjection $\pi: \tilde{A} \rightarrow A$ in \mathfrak{Art} factors as a finite composition of small extensions.

Proof. Consider the extensions $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, and the vector subspaces $I_r := \{a \in I \mid a \cdot \mathfrak{m}_A^r = 0\}$. We have $\mathfrak{m}_A \cdot I_r \subseteq I_{r-1} \subseteq I_r$, and we have an N such that $\mathfrak{m}_A^N = 0$. So we can consider

$$\tilde{A} = \tilde{A}/I_0 \rightarrow \tilde{A}/I_1 \rightarrow \tilde{A}/I_2 \rightarrow \dots \rightarrow \tilde{A}/I_N = A.$$

It is easy to check that each step is a semismall extension. To go from semismall extensions to small one, we use the following exercise. \square

1.16 EXERCISE. If $\tilde{A} \rightarrow A$ is a semismall extension, then all vector subspaces of I are ideals in \tilde{A} .

1.17 COROLLARY. Let $F \rightarrow G$ be a natural transformation of functors from \mathfrak{Art} to \mathfrak{Sets} . The following are equivalent:

1. for every $\tilde{A} \rightarrow A$, the map $F(\tilde{A}) \rightarrow F(A) \times_{G(A)} G(\tilde{A})$ is surjective (bijective);
2. 1 holds for every square zero extension;
3. 1 holds for every semismall extension;
4. 1 holds for every small extension;

Before proving the corollary, let us explain the statement with an example. Let $\varphi: X \rightarrow Y$ be a morphism of schemes, $F := h_X$ and $G := h_Y$; then the map in $\mathbf{1}$ becomes

$$\{\tilde{f}: \text{Spec } \tilde{A} \rightarrow X\} \rightarrow \left\{ \left(\begin{array}{l} f: \text{Spec } A \rightarrow X \\ \tilde{g}: \text{Spec } \tilde{A} \rightarrow Y \end{array} \right) \left| \begin{array}{ccc} \text{Spec } A & \xrightarrow{f} & X \\ i \downarrow & & \downarrow \varphi \\ \text{Spec } \tilde{A} & \xrightarrow{\tilde{g}} & Y \end{array} \right. \right\};$$

saying that $\tilde{f} \mapsto (f, \tilde{g})$ means that $f = \tilde{f} \circ i$ and $\tilde{g} = \varphi \circ \tilde{f}$, i.e. that \tilde{f} commutes with the diagram. Asking that the map is surjective (bijective) means asking that a lifting exists (exists and is unique).

Proof. We need to prove that if we know $\mathbf{1}$ for small extensions, then we know it for any extension. We prove this by induction on the dimension d of I as a K -vector space. If $d = 1$, $\tilde{A} \rightarrow A$ is small, so there is nothing to prove. If we know $\mathbf{1}$ for any extension with dimension of the kernel less than d , and $\tilde{A} \rightarrow A$ has $\dim_K I = d$, then we can find a factorization $\tilde{A} \rightarrow A_1 \xrightarrow{\text{small}} A$.

Having this, consider $(f, \tilde{g}) \in F(A) \times_{G(A)} G(\tilde{A})$; let $g_1 \in G(A_1)$ be the image of \tilde{g} ; by smallness, there exists (exists and is unique) $f_1 \in F(A_1)$ mapping to (f, g_1) . Consider $(f_1, \tilde{g}) \in F(A_1) \times_{G(A_1)} G(\tilde{A})$; by induction there exists (exists and is unique) \tilde{f} mapping to (f_1, \tilde{g}) . \square

1.18 EXERCISE. Do the proof of the corollary in the geometric setup.

2 DEFORMATIONS OF MORPHISMS

2.1 Liftings

Lecture 2 (1 hour)
August 31st, 2010

Let $\pi: \tilde{A} \rightarrow A$ be a surjective homomorphism of rings with $I := \ker \pi$ square zero; let also $\psi: R \rightarrow A$ be any ring homomorphism and $\tilde{\psi}_0$ a lifting of ψ , in the sense that we have the following diagram:

$$\begin{array}{ccc} & \tilde{A} & \\ \tilde{\psi}_0 \nearrow & & \downarrow \pi \\ R & \xrightarrow{\psi} & A. \end{array}$$

2.1 LEMMA. In this situation, there is a natural action of $\text{Der}(R, I)$ on the set of liftings $\tilde{\psi}: R \rightarrow \tilde{A}$ that is simply transitive (in other words, it makes the set into a torsor under $\text{Der}(R, I)$).

Proof. Since we assume there is a specific lifting $\tilde{\psi}_0$, we know that the set is nonempty. Let $\tilde{\psi}: R \rightarrow \tilde{A}$ be any map (of sets) and define $\lambda := \tilde{\psi} - \tilde{\psi}_0$. To say that $\tilde{\psi}$ is a lifting of ψ is equivalent to say that:

1. $\tilde{\psi}$ is a group homomorphism;
2. $\pi \circ \tilde{\psi} = \psi$;
3. $\tilde{\psi}(xy) = \tilde{\psi}(x)\tilde{\psi}(y)$.

We can express these conditions in terms of λ :

1. $\lambda: R \rightarrow \tilde{A}$ is a group homomorphism;
2. $\pi \circ \lambda = 0$, i.e., $\lambda: R \rightarrow I \subseteq \tilde{A}$;
3. since I is square zero, $\lambda(xy) = \lambda(x)\tilde{\psi}_0(y) + \lambda(y)\tilde{\psi}_0(x)$, and since $\lambda(x) \in I$, this is equivalent to $\lambda(xy) = \lambda(x)\psi(y) + \lambda(y)\psi(x)$.

These three properties tells us exactly that $\tilde{\psi}$ is a lifting if and only if λ is a derivation. \square

2.2 REMARK. Use the same assumptions on π , and assume the diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \tilde{A} \\ \downarrow & & \downarrow \pi \\ R & \xrightarrow{\psi} & A \end{array}$$

commutes. Then the set of $\tilde{\psi}: R \rightarrow \tilde{A}$ that commutes with the diagrams is either empty or a torsor under $\text{Der}_S(R, I)$. Note that we have isomorphisms

$$\text{Der}_S(R, I) \cong \text{Hom}_R(\Omega_{R/S}, I) \cong \text{Hom}_A(\Omega_{R/S} \otimes_R A, I).$$

2.3 REMARK. Assume now $A, \tilde{A} \in \mathfrak{Art}$ and π is semismall. Then A has a natural map to K that induces a K -valued point on $\text{Spec } R$, and the diagram becomes:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \tilde{A} \\ \downarrow & & \downarrow \pi \\ R & \xrightarrow{\psi} & A \\ \downarrow & & \downarrow \\ K & \xlongequal{\quad} & K. \end{array}$$

In this case, the set of liftings, when it is nonempty, is a torsor under

$$\text{Hom}_A(\Omega_{R/S} \otimes_R A, I) \cong \text{Hom}_K(\Omega_{R/S} \otimes_R K, I).$$

If $X := \text{Spec } R$, $Y := \text{Spec } S$, and $x \in X$ is the specified K -valued point, then the group is

$$\text{Hom}(T_{x/Y}^*(x), I) = T_{x/Y}(x) \otimes I.$$

In order to justify the first part of this remark, a square zero extension would have been enough. We asked for a semismall extension because when we restrict to these, the result ends up not depending on the specific extension, but only on the morphism $X \rightarrow Y$ and on the point x .

2.4 REMARK. These reasoning works in greater generality. For example, let $T \hookrightarrow \tilde{T}$ be a closed embedding of schemes with $\mathcal{I}_{\tilde{T}/T}^2 = 0$. Let

$$\begin{array}{ccc} T & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \tilde{T} & \xrightarrow{g} & Y \end{array}$$

be a commutative diagram of schemes; then the set of morphisms $\tilde{f}: \tilde{T} \rightarrow X$ commuting with the diagram is either empty or a torsor under $\text{Hom}(f^*\Omega_{X/Y}, \mathcal{I}_{T/\tilde{T}})$.

2.2 Existence of a lifting

Let $p: X \rightarrow Y$ be a morphism of affine schemes, and $\tilde{A} \rightarrow A$ a semismall extension in $\mathfrak{A}rt$ with kernel I . Assume given a commutative diagram

$$\begin{array}{ccc} \text{Spec } A & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ \text{Spec } \tilde{A} & \xrightarrow{g} & Y \end{array}$$

Then we know that the set of arrows $\tilde{f}: \text{Spec } \tilde{A} \rightarrow X$ commuting with the diagram is empty or a torsor; how do we know which one of the two it is?

Assume that $X := \text{Spec } R$ and $Y := \text{Spec } S$; then if $P := S[x_1, \dots, x_n]$, we have $R = P/J$; moreover $\mathbb{A}_Y^n = \mathbb{A}^n \times Y = \text{Spec } P$. Then a lifting h as in

$$\begin{array}{ccc} \text{Spec } A & \xrightarrow{f} & X \\ \downarrow & \nearrow h & \downarrow \\ \text{Spec } \tilde{A} & \xrightarrow{g} & Y \end{array} \quad \begin{array}{c} \mathbb{A}_Y^n \\ \downarrow \end{array}$$

always exists: we just have to look at the diagram of rings

$$\begin{array}{ccc} A & \xleftarrow{\varphi} & P/J \\ \uparrow & & \uparrow \\ \tilde{A} & \xleftarrow{\psi} & S \end{array} \quad \begin{array}{c} P \\ \uparrow \\ S \end{array} \quad \begin{array}{c} \alpha \\ \nearrow \end{array}$$

and work out explicitly the map α . Observe that, when restricted to J , such a homomorphism α takes values in I . This is so because $J \subseteq \ker \pi \circ \alpha$, so $\alpha(J) \subseteq \ker \pi = I$. Also, even if h does not need to factor through X , we can see that if it does, it factors in a unique way. In the following fix a specific lifting α_0 .

Consider the set of all liftings $\{\alpha\}$. In it, the set of all liftings to X injects (it consists of the α with $\alpha|_J = 0$). Moreover, from what we said, $\{\alpha\}$ has the structure of a torsor under $\text{Hom}(\Omega_{P/S}, I) \cong \text{Der}_S(P, I)$.

Consider the sequence

$$\text{Hom}(\Omega_{P/S}, I) \rightarrow \text{Hom}(J, I) \rightarrow \text{coker} \rightarrow 0$$

where the first homomorphism send a derivation $\varphi: P \rightarrow I$ to its restriction to J ; then a lifting to X exists if and only if $\varphi|_J + \alpha_0|_J = 0$ for some φ , that is, if and only if $\alpha_0|_J$ is sent to 0 in the cokernel.

We have to understand what this cokernel is. First of all, note that since π is semismall, J^2 maps to zero in I and so $\text{Hom}(J, I) = \text{Hom}(J/J^2, I)$. Also, we can manipulate algebraically the sequence to obtain

$$(2) \quad \text{Hom}_A(\Omega_{P/S} \otimes_P A, I) \rightarrow \text{Hom}_A(J/J^2 \otimes_P A, I) \rightarrow \text{coker} \rightarrow 0.$$

Then we proceed with a special case, namely we assume that $X \rightarrow Y$ is smooth. In this case, we have the exact sequence

$$0 \rightarrow J/J^2 \rightarrow \Omega_{\mathbb{A}_Y^n/Y}|_X \rightarrow \Omega_{X/Y} \rightarrow 0.$$

Pulling back via f and applying $\text{Hom}(\bullet, I)$, it becomes

$$\text{Hom}(f^*\Omega_{\mathbb{A}_Y^n/Y}|_X, I) \rightarrow \text{Hom}(f^*J/J^2, I) \rightarrow \text{Ext}_{\text{Spec } A}^1(f^*\Omega_{X/Y}, I).$$

This correspond precisely to the sequence (2); moreover, in this case the Ext group (hence the cokernel of the sequence (2)) is zero, because $\Omega_{X/Y}$ is locally free. We have obtained the following.

2.5 COROLLARY. *If $p: X \rightarrow Y$ is a smooth morphism of affine schemes, the lifting always exists for any square zero extension (not necessarily in \mathfrak{Art}). Moreover, if the square zero extension is in \mathfrak{Art} , we need not to assume that X and Y are affine.*

Consider now $R := P/J$ and $X := \text{Spec } R$. Fix $x \in X$ a point, that is, fix a maximal ideal $\mathfrak{m}_x \subseteq R$. Infact, we may assume $\mathfrak{m}_x = ((x_1, \dots, x_n), \mathfrak{m}_y)$, with $\mathfrak{m}_y \subseteq S$. All that counts now is the morphism $\hat{S} \rightarrow \hat{R}$ of the formal completions at \mathfrak{m}_x and \mathfrak{m}_y respectively. Note that $\hat{R} = \hat{S}_{[[x_1, \dots, x_n]]}/\hat{J}$. We can simplify this: for every $f \in \hat{J}$, we can split f as $f_0 + f_1 + \dots$; since f is in the maximal ideal \mathfrak{m}_x , $f_0 \in \mathfrak{m}_y$. Also, $f_1 = \sum_i a_i x_i$, and every time $f_0 = 0$ and there is an a_i which is a unit, we can use f_1 to get rid of a variable. This is the formal series counterpart of the fact that whenever the first derivative of a function does not vanish, we can solve for a variable.

Using this procedure, and assuming to work locally in the étale topology,

we can assume $R = S[x_1, \dots, x_n]/J$ and that the derivatives of the elements in J are all zero. So the map $J \rightarrow \Omega_{A^0/Y}$ is the zero map and the obstruction space is $(I/mI)^\vee \otimes I$.

Summing up: if $p: X \rightarrow Y$ is a morphism of schemes and $x \in X$ is a point with $p(x) = y$, and we consider $p: h_{X,x} \rightarrow h_{Y,y}$ the natural transformation of functors $\mathfrak{Art} \rightarrow \mathfrak{Sets}$, then for every semismall extension $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, and for every $(\varphi, \tilde{\psi}) \in h_{X,x}(A) \times_{h_{Y,y}(A)} h_{Y,y}(\tilde{A})$ there is an obstruction to lift $(\varphi, \tilde{\psi})$ to $h_{X,x}(\tilde{A})$ which is an element of $(I/mI)^\vee \otimes I$. If the obstruction space vanishes, the set of liftings is a torsor under $T_{X/Y}(x) \otimes I$. In other words, there is an exact sequence of groups and sets

$$0 \rightarrow T_{X/Y}(x) \otimes_K I \rightarrow h_{X,x}(\tilde{A}) \rightarrow h_{X,x}(A) \times_{h_{Y,y}(A)} h_{Y,y}(\tilde{A}) \rightarrow (I/mI)^\vee \otimes I.$$

2.6 DEFINITION. We say that the sequence of groups and sets is *exact* if and only if:

1. the action of the group $T_{X/Y}(x) \otimes I$ is free on $h_{X,x}(\tilde{A})$;
2. two points of $h_{X,x}(\tilde{A})$ have the same image if and only if they are in the same orbit;
3. an element of $h_{X,x}(A) \times_{h_{Y,y}(A)} h_{Y,y}(\tilde{A})$ goes to zero if and only if it has a lifting.

2.3 Tangent and obstruction spaces

2.7 DEFINITION. Let $F \rightarrow G$ be a natural transformation of functors $\mathfrak{Art} \rightarrow \mathfrak{Sets}$. We say that two vector spaces T^1 and T^2 are *tangent and obstruction spaces* for $F \rightarrow G$ if for every semismall extension $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, there is an exact sequence of groups and sets

$$0 \rightarrow T^1 \otimes I \rightarrow F(\tilde{A}) \rightarrow F(A) \times_{G(A)} G(\tilde{A}) \rightarrow T^2 \otimes I$$

which is functorial, that is, for any other semismall extension $0 \rightarrow I_1 \rightarrow \tilde{A}_1 \rightarrow A_1 \rightarrow 0$ with morphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & \tilde{A} & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & I_1 & \longrightarrow & \tilde{A}_1 & \longrightarrow & A_1 \longrightarrow 0, \end{array}$$

the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^1 \otimes I & \longrightarrow & F(\tilde{A}) & \longrightarrow & F(A) \times_{G(A)} G(\tilde{A}) \longrightarrow T^2 \otimes I \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T^1 \otimes I_1 & \longrightarrow & F(\tilde{A}_1) & \longrightarrow & F(A_1) \times_{G(A_1)} G(\tilde{A}_1) \longrightarrow T^2 \otimes I_1 \end{array}$$

2. DEFORMATIONS OF MORPHISMS

commutes.

Lecture 3 (1 hour)
September 1st, 2010

Note that, even if it is not part of the definition, most of the time we will assume $F(K) = G(K) = \{\text{pt}\}$.

2.8 EXAMPLE. Let $F := h_{X,x}$, $G := h_{Y,y}$, and $F \rightarrow G$ induced by $p: X \rightarrow Y$ with $y = p(x)$. Then the tangent space T^1 is $T_{X/Y}(x)$ and T^2 is the cokernel we saw before.

2.9 REMARK. Let $F: \mathcal{A}rt \rightarrow \mathcal{S}ets$ be a functor. We can make absolute the previous relative definition defining tangent and obstruction spaces of F to be tangent and obstruction spaces for $F \rightarrow \text{pt}$, where pt is the functor $h_{\text{Spec } K}$.

2.10 REMARK. Let $p: F \rightarrow G$ be a natural transformation and assume $G(K) = \{\text{pt}\}$. Let $A \in \mathcal{A}rt$, then we have a sequence $K \rightarrow A \rightarrow K$ and applying G we get $G(K) \rightarrow G(A) \rightarrow G(K)$. Let F_0 be the fiber of p , so that

$$F_0(A) = \{a \in F(A) \mid p(a) = \text{pt} \in G(A)\}.$$

One can prove that $F_0: \mathcal{A}rt \rightarrow \mathcal{S}ets$ is a functor and T^1 and T^2 for p are T^1 and T^2 also for F_0 .

We have not specified if tangent and obstruction space are unique.

The second is trivially not unique, since the only property we want for T^2 is that an object maps to 0 if and only if it has a lifting; so for any inclusion $T^2 \subseteq V^2$, V^2 is also an obstruction space. Therefore, the best hope for having a unique T^2 is to define a minimal obstruction space T_{\min}^2 such that for every other obstruction space T^2 , we have a unique compatible morphism $T_{\min}^2 \hookrightarrow T^2$. This is very reasonable theoretically, but in practice finding this minimal obstruction space is very difficult and doable only when it is given as a cokernel as in the example.

As for the tangent space, we note that T^1 , as a vector space, is determined by $F \rightarrow G$; indeed is determined by F_0 , because we have the exact sequence

$$0 \rightarrow T^1 \otimes I \rightarrow F_0(\tilde{A}) \rightarrow F_0(A) \rightarrow T^2 \otimes I,$$

and the only point in $F(K)$ maps to a liftable element in $F_0(A)$ for all A . Hence we can prove that, as a set, $T^1 = F_0(K[\varepsilon]/\varepsilon^2)$. Let $D := K[\varepsilon]/\varepsilon^2$; then the multiplication by $\lambda \in K$ is induced by the morphism $D \rightarrow D$ sending ε to $\lambda\varepsilon$. For the addition, we have to consider $D_2 := K[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_1\varepsilon_2, \varepsilon_2^2)$ and prove that the map $F_0(D_2) \rightarrow F_0(D) \times F_0(D)$ induced by the two maps $D_2 \rightarrow D$ sending $(\varepsilon_1, \varepsilon_2)$ respectively to $(\varepsilon, 0)$ and $(0, \varepsilon)$ is a bijection; so we have

$$\begin{array}{ccc} F_0(D_2) & \longrightarrow & F_0(D) \times F_0(D) \\ \downarrow & \swarrow & \\ F_0(D) & & \end{array}$$

where the vertical arrow is induced by $(\varepsilon_1, \varepsilon_2) \mapsto (\varepsilon, \varepsilon)$ and the diagonal is the

addition we wanted.

This construction applies also to the case $F_0 = \mathfrak{h}_{X,x}$, where $T_x X$ is the set of morphisms $\text{Spec } D \rightarrow X$ sending the closed point to x , equipped with the multiplication and addition we defined.

2.11 EXAMPLE. Let C, B, X be schemes, $G := \mathfrak{h}_{B,0}$, and assume we have the morphisms

$$\begin{array}{ccc} C & & X \\ \text{flat} \searrow & & \swarrow \text{smooth} \\ & B & \end{array} \quad \begin{array}{ccc} C_0 & \xrightarrow{f_0} & X_0 \\ \searrow & & \swarrow \\ & \{0\} & \end{array}$$

Define

$$F(A) := \{(\varphi, f) \mid \varphi \in \mathfrak{h}_{B,0}(A), f: C_A \rightarrow X_A, f|_{C_0} = f_0\}$$

where $C_A = C \times_B \text{Spec } A$, $X_A = X \times_B \text{Spec } A$ and f is defined over $\text{Spec } A$. Let $p: F \rightarrow G$ be the forgetful transformation sending (φ, f) to φ . If C and X are affine over B , then C_A and X_A are affine; in this case, $T^2 = 0$ and $T^1 = \Gamma(C_0, f^* T_{X_0})$. In the general case, we cover X by affines V_j and C by affines U_j so that $f_0(U_j \cap C_0) \subseteq V_j \cap X_0$. The important thing to notice is that C_A has the same topology as C , so $C_A \cap U_j$, $X_A \cap V_j$ are also affines, and similarly for \tilde{A} . Let $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ be a semismall extension and consider an element $(\varphi, f, \tilde{\varphi}) \in F(A) \times_{G(A)} G(\tilde{A})$. This induces the diagram

$$\begin{array}{ccc} C_0 & \xrightarrow{f_0} & X_0 \\ \downarrow & & \downarrow \\ C_A & \xrightarrow{f} & X_A \\ \downarrow & & \downarrow \\ C_{\tilde{A}} & & X_{\tilde{A}} \\ \downarrow & & \downarrow \\ \text{Spec } \tilde{A} & = & \text{Spec } \tilde{A} \end{array}$$

and we ask for the existence of a morphism $\tilde{f}: C_{\tilde{A}} \rightarrow X_{\tilde{A}}$ fitting in the diagram. Cover $C_{\tilde{A}}$ with $C_{\tilde{A}} \cap U_j$; then by the previous case we can find \tilde{f}_j extending $f|_{C_A \cap U_j}$. Do these local data glue? On $C_{\tilde{A}} \cap U_{i,j}$, \tilde{f}_i and \tilde{f}_j have the same restriction to $C_A \cap U_{i,j}$, so they differ by $\vartheta_{i,j} \in \Gamma(U_{i,j}, f_0^* T_{X_0}) \otimes I$; it is easy to see that these defines a 1-cocycle with values in $f_0^* T_{X_0} \otimes I$. But we did a choice selecting the \tilde{f}_j ; changing this data, one sees that the the cocycle changes by a coboundary, so that $T^i = H^{i-1}(X_0, f^* T_{X_0})$ for $i \in \{1, 2\}$. Note that the flatness is used to know that the kernel of $\mathcal{O}_{C_{\tilde{A}}} \rightarrow \mathcal{O}_{C_A} \rightarrow 0$ is $F \otimes_K \mathcal{O}_{C_0}$.

3. DEFORMATIONS OF A SCHEME

We did not assume anything about B ; indeed, the same argument apply when B is an algebraic stack. In particular if we fix a dimension d and integers $n, g \geq 0$, we can take $B := \overline{M}_{g,n} \times \text{Var}_d$, where $\overline{M}_{g,n}$ is the moduli stack of nodal, connected, arithmetic genus g curve, with n distinct smooth marked points, and Var_d is the stack of smooth projective varieties of dimension d . The fact that deformation works over the former is the key to define Gromov-Witten invariants, while the one that it works over the latter is the key point to show that they are deformation invariants.

2.12 EXAMPLE. The obstruction space obtained using cohomology is not always minimal. Indeed, consider a smooth surface X_0 , $C_0 := \mathbb{P}^1$, and a closed embedding $f_0: C_0 \rightarrow X_0$ that is the inclusion of a (-2) -curve, so that $N_{C_0/X_0} = \mathcal{O}_{\mathbb{P}^1}(-2)$. We have the exact sequence

$$0 \rightarrow T_{C_0} \rightarrow T_{X_0}|_{C_0} \rightarrow N_{C_0/X_0} \rightarrow 0$$

and this yields isomorphisms

$$\begin{aligned} K^3 &\cong H^0(C_0, T_{C_0}) \rightarrow H^0(C_0, f_0^* T_{X_0}) = T^1, \text{ and} \\ T^2 &= H^1(f_0^* T_{X_0}) \rightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \cong K. \end{aligned}$$

Let $B := \text{Spec } K$; we can define a map $\text{Aut}(\mathbb{P}^1) \rightarrow \text{Mor}(C_0, X_0)$ sending g to $f_0 \circ g$; this is clearly injective and, since the source is smooth of dimension 3 and the target has tangent space of dimension 3, it has to be an isomorphism with a connected component of $\text{Mor}(C_0, X_0)$. In particular, $\text{Mor}(C_0, X_0)$ is smooth at f_0 .

There often exists a deformation

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \text{smooth, projective} \\ 0 & \hookrightarrow & B \end{array}$$

such that for $b \in B$ general, the fiber X_b contains no (-2) -curve.

A similar case is then V_0 is a vector bundle over X_0 , $G = \mathfrak{h}_{B,0}$ and we define $F(A)$ to be the set of pairs (φ, V_A) with V_A a vector bundle over $X \times_B \text{Spec } A$ extending V_0 and $\varphi \in \mathfrak{h}_{B,0}$. In this case, $T^i = H^i(X_0, \text{End } V_0)$ for $i \in \{1, 2\}$.

3 DEFORMATIONS OF A SCHEME

3.1 Definition

We could define a deformation of a scheme X_0 over $\text{Spec } A$ as a flat map $X_A \rightarrow \text{Spec } A$ such that $X_A \times_{\text{Spec } A} \text{Spec } K \cong X_0$. This has some problems, indeed the usual definition definition is slightly different: a deformation of X_0 over $\text{Spec } A$ is a flat map $X_A \rightarrow \text{Spec } A$ together with an isomorphism

$X_0 \xrightarrow{\sim} X_A \times_{\text{Spec } A} \text{Spec } K$, or equivalently a cartesian diagram

$$\begin{array}{ccc} X_0 & \longrightarrow & X_A \\ \downarrow & & \downarrow \\ \text{Spec } K & \longrightarrow & \text{Spec } A \end{array}$$

3.1 LEMMA. *If $X_0 = \text{Spec } R$ is smooth and X_A is a deformation of X_0 over $\text{Spec } A$, then X_A is trivial, i.e.*

$$\begin{array}{ccccc} & \xrightarrow{\text{natural inclusion}} & & & \\ X_0 & \hookrightarrow & X_A & \xrightarrow{\sim} & X_0 \times \text{Spec } A \\ & & \downarrow & \swarrow & \\ & & \text{Spec } A. & & \end{array}$$

The proof of this lemma can be found in Artin's "Lectures on deformations of singularities" (1974).

3.2 Extending a deformation

Let $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ be a semismall extension. The aim is to understand when we have a double cartesian square

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_A & \longrightarrow & X_{\tilde{A}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Spec } A & \longrightarrow & \text{Spec } \tilde{A} \end{array}$$

i.e., when we can extend the deformation over $\text{Spec } A$ to a deformation over \tilde{A} and how different deformations are related.

Consider an open and affine cover $\{U_i\}$ of X_0 ; there is an induced cover of X_A , namely $\{U_i \cap X_A\}$. By Lemma 3.1, for every i there exists an isomorphism $\varphi_i: X_A \cap U_i \xrightarrow{\sim} U_i \times \text{Spec } A$, inducing the identity on the reduced structure. On double intersections, there are two isomorphisms $U_{i,j} \cap X_A \rightarrow U_{i,j} \times \text{Spec } A$, so we can define automorphisms

$$\varphi_{i,j} = \varphi_j \circ \varphi_i^{-1}: U_{i,j} \times \text{Spec } A \rightarrow U_{i,j} \times \text{Spec } A.$$

One can check that on triple intersection, the $\varphi_{i,j}$ satisfy the cocycle condition, i.e. $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$. We can then reconstruct completely X_A from the data of the $\varphi_{i,j}$.

Now, if we assume to have a compatible deformation $X_{\tilde{A}}$ over \tilde{A} , by Lemma 3.1 we can extend φ_i to $\tilde{\varphi}_i: U_i \cap X_{\tilde{A}} \rightarrow U_i \times \text{Spec } \tilde{A}$. We can also extend $\varphi_{i,j}$, in such a way that the cocycle condition holds. The converse is also true: if we can extend the $\varphi_{i,j}$ to some $\tilde{\varphi}_{i,j} \in \text{Aut}(U_{i,j} \times \text{Spec } \tilde{A}/\tilde{A})$ that satisfy the cocycle condition, then we can define $X_{\tilde{A}}$ by glueing.

To prove this claim, consider the diagram

$$\begin{array}{ccc} U_{i,j} \times \text{Spec } A & \xrightarrow{\varphi_{i,j}} & U_{i,j} \\ \downarrow & \nearrow \tilde{\varphi}_{i,j} & \\ U_{i,j} \times \text{Spec } \tilde{A} & & \end{array}$$

where the $\tilde{\varphi}_{i,j}$ exist because the source is affine and the target is smooth. Then we can choose $\tilde{\varphi}_{i,j}$ in a random way, and we have to check that the cocycle condition holds on $U_{i,j,k} \times \text{Spec } \tilde{A}$. The three maps $\tilde{\varphi}_{i,k}$, $\tilde{\varphi}_{j,k}$, and $\tilde{\varphi}_{i,j}$ agree on $U_{i,j,k} \times \text{Spec } A$, hence they differ by $\vartheta_{i,j,k} \in \Gamma(U_{i,j,k}, T_{X_0} \otimes I)$. It is easy to check that these elements defines a Čech 2-cocycle on the cover $\{U_{i,j,k}\}$ with values in $T_{X_0} \otimes I$. When we make a different choice of $\tilde{\varphi}_{i,j}$, the cocycle $\vartheta_{i,j,k}$ changes by a coboundary.

We obtain that $X_{\tilde{A}}$ exists if and only if $[\vartheta] \in H^2(X_0, T_{X_0}) \otimes I$ is zero. Moreover, if $[\vartheta] = 0$ for a choice $\tilde{\varphi}_{i,j}$, then any other choice differ by a 1-cocycle (and gives the same deformation if they differ by a 1-coboundary), so the set of all possible choice (if nonempty) is a torsor under $H^1(X_0, T_{X_0}) \otimes I$. Indeed, we have the following.

3.2 LEMMA. *Assume $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ is a semismall extension and*

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_A \\ \downarrow & & \downarrow \pi \text{ (flat)} \\ \text{Spec } K & \rightarrow & \text{Spec } A \end{array}$$

is a cartesian diagram. Then, if there exists a cartesian diagram

$$\begin{array}{ccc} X_A & \xrightarrow{j} & X_{\tilde{A}} \\ \downarrow & & \downarrow \tilde{\pi} \\ \text{Spec } A & \rightarrow & \text{Spec } \tilde{A}, \end{array}$$

the set of triples $(X_{\tilde{A}}, \tilde{\pi}, j)$ as before, modulo isomorphisms, is a torsor under $H^1(X, T_{X_0}) \otimes I$. Here an isomorphism $(X_{\tilde{A}}, \tilde{\pi}, j) \rightarrow (X'_{\tilde{A}}, \tilde{\pi}', j')$ is an isomorphism $\psi: X_{\tilde{A}} \rightarrow X'_{\tilde{A}}$ commuting with the maps $j, j', \tilde{\pi}, \tilde{\pi}'$.

3.3 Deformation functor of a scheme

Consider the functor $\text{Def}_{X_0}: \mathfrak{Art} \rightarrow \mathfrak{Sets}$ defined by letting $\text{Def}_{X_0}(A)$ be the set of triples (X_A, π, i) with a cartesian diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_A \\ \downarrow & & \downarrow \pi \text{ (flat)} \\ \text{Spec } K & \rightarrow & \text{Spec } A \end{array}$$

modulo isomorphisms. Then for every semismall exact sequence $0 \rightarrow I \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, there is an exact sequence of groups and sets

$$H^1(X_0, T_{X_0}) \otimes I \rightarrow \text{Def}_{X_0}(\tilde{A}) \rightarrow \text{Def}_{X_0}(A) \rightarrow H^2(X_0, T_{X_0}) \otimes I.$$

The reason why there is no 0 on the left is the following. If $\tilde{\xi} := (X_{\tilde{A}}, \tilde{\pi}, j) \in \text{Def}_{X_0}(\tilde{A})$, then the stabilizer of $\tilde{\xi}$ inside $H^1(X_0, T_{X_0}) \otimes I$ gives an exact sequence

$$0 \rightarrow H^0(X, T_{X_0}) \otimes I \rightarrow \text{Aut}(\tilde{\xi}) \rightarrow \text{Aut}(\xi) \rightarrow \text{Stab}(\tilde{\xi}) \rightarrow 0,$$

where ξ is the image of $\tilde{\xi}$ in $\text{Def}_{X_0}(A)$.

3.4 Groupoids, stacks and 2-functors

The problem here is that we modded out isomorphisms in the definition of Def_{X_0} ; the idea of Artin instead is to remember isomorphisms (and in particular automorphisms), in this way: define $\text{Def}_{X_0}(A)$ to be the set of triples (X_A, π, i) together with the set of isomorphisms between them. This object is no more a set, but a groupoid.

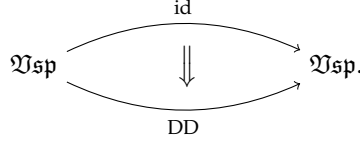
3.3 DEFINITION. A *groupoid* is a category such that all morphisms are isomorphisms.

Hence the new Def_{X_0} is a pseudofunctor $\mathfrak{Art} \rightarrow \mathfrak{Groupoids}$. Here we said pseudofunctor because $\mathfrak{Groupoids}$ has a natural structure of 2-category.

3.4 EXAMPLE. The easiest example of a groupoid is a category where we canceled all non invertible morphisms.

Recall that we are always working over an algebraically closed field K . Consider \mathfrak{Vsp} , the category of K -vector spaces; we can define two functors between \mathfrak{Vsp} and itself, the identity and the double dual functor. There exists a natural transformation between them that is a natural equivalence when restricted to finite dimensional vector spaces. Usually, one draws such a situ-

ation with this kind of diagrams:



In these diagrams, one has not only vertices and arrows (objects and morphisms), but also faces (natural transformations).

So, when we have a natural structure $F: \mathfrak{A}rt \rightarrow \mathfrak{S}ets$, we have two vector spaces T_F^1 and T_F^2 (tangent and obstructions); when the natural structure for our deformation functor is $F: \mathfrak{A}rt \rightarrow \mathfrak{G}roupoids$, we obtain three vector spaces: in addition to the previous one we have T_F^0 that represent the tangent space to the automorphisms group of the (only) point $F(K)$. This development was in some sense hinted by deformation theory itself, whose result showed that there was still to give an interpretation to the cohomology group $H^0(X, T_{X_0})$.

This intuition took Artin to define his version of algebraic stacks; already Deligne and Mumford defined algebraic stacks starting from functors to the 2-category of groupoids, but their version is too soft: the way one recognize DM-stacks amongst Artin stacks is precisely to look at them infinitesimally and see that the tangent space to the automorphisms group is 0.

In the case of the deformations of a scheme X_0 , the tangent space $T_{id} \text{Aut}(X_0)$ is (by what we said in the previous lectures) $H^0(X_0, id^* T_{X_0})$, that is $H^0(X_0, T_{X_0})$.

If $H^0(X_0, T_{X_0}) = 0$, we can prove by induction on $\dim_K A$ that $\text{Aut}(\xi) = 0$ for every ξ . Moreover, in many occasion where $\text{Aut}(\tilde{\xi})$ does not change too wildly, one still has $\text{Stab}(\tilde{\xi}) = 0$.

3.5 Cotangent complex

One modern viewpoint is that for every morphism of schemes $f: X \rightarrow Y$, there is a simplicial (in characteristic 0, dg) resolution $\varepsilon: X \hookrightarrow \tilde{X}$ in an appropriate derived category, such that ε is a quasi-isomorphism and induces a unique (up to quasi isomorphisms) quasi smooth $\tilde{f}: \tilde{X} \rightarrow Y$. This leads to this program to solve a problem:

- solve the problem in the smooth case;
- express the solution in terms of the cotangent bundle;
- replace the cotangent bundle $\Omega_{X/Y}$ with the cotangent complex $\varepsilon^* \Omega_{\tilde{X}/Y} = \mathbb{L}_{X/Y}^\bullet \in D_{\text{coh}}^{\leq 0}(X)$; this has the following properties:
 1. $\tau_{\geq -1} \mathbb{L}_{X/Y}^\bullet = [I/I^2 \rightarrow \Omega_{W|X}]$ for W a variety that has a closed embedding $X \hookrightarrow W$ and a smooth morphism $W \rightarrow Y$;
 2. $\mathbb{L}_{X/Y}^\bullet = \tau_{\geq -1} \mathbb{L}_{X/Y}^\bullet$ if $X \rightarrow Y$ is lci;
 3. it is functorial.

3.5 EXERCISE. Let $Y := \text{Spec } K$, X a nodal curve (since everything is étale local, we can think of $X := \text{Spec } K[x, y]/xy$); the variety W can be taken to be \mathbb{A}^2 . Compute $\mathbb{L}_{X/Y}^\bullet = \Omega_{X/Y}$ and prove

$$\mathcal{E}xt^2(\Omega_{X/Y}, \mathcal{O}_X) = 0, \text{ and}$$

$$\mathcal{E}xt^1(\Omega_{X/Y}, \mathcal{O}_X) = \mathcal{O}_p,$$

where p is the node. This is the key point to deduce the deformation theory of nodal curves.