# Moduli of representations of quivers
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## Contents



## 1 Why moduli of quivers representations?

*Lecture 1 (1 hour)*
*June 11$^{th}$, 2012*

First of all, quivers representations are a way to formalize problems in linear algebra, typically very classic. Often these end up with a classification of all solutions, that depends on continuous and discrete parameters. Moduli spaces are exactly a way to materialize the continuous parameters.

Even if one is interested in completely different moduli problems, for example moduli of vector bundles, quite often can find that moduli of quivers representations are related to these problems, for example being models (not intended in a formal sense), or toy examples, or part of the solution, etc.

Another motivation is that moduli of quivers representations are simple enough to be used as testing ground for techniques in moduli theory.

Finally, in one direction of non-commutative algebraic geometry, moduli of quivers representations appear as "commutative shadows" of smooth, non-commutative algebraic geometry.

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## 2 Representations of quivers

2.1 definition. A *quiver* is a finite oriented graph $Q$; in particular we will denote with $Q_0$ the set of vertices, with $Q_1$ the set of edges and we will write $\alpha\colon i \to j$ for an edge. The maps $s, t\colon Q_1 \to Q_0$ associate to each edge the source and the target vertex.

2.2 example. We do not rule out the possibility of having loops, or multiple edges between the same source and target, or even multiple loops. We just assume that the sets $Q_0$ and $Q_1$ are finite.

Intuitively, a representation of a quivers consists of fixing a vector space for every vertex of the quiver and a linear map for every edge.

2.3 definition. Let $k$ be a field and $Q$ a quiver. A *$k$-representation of $Q$* consists of a finite dimensional $k$-vector space $V_i$ at each vertex $i \in Q_0$ and a $k$-linear map $V_\alpha\colon V_i \to V_j$ for every edge $\alpha\colon i \to j$.

2.4 example. If we have the quiver

$$\bullet \longrightarrow \bullet$$

then studying the representation of it consists of studying all maps between vector spaces up to base change in the source or the target. If the quiver is

$$\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$$

then we are studying linear self-maps of vector spaces up to a common base change in source and target.

In the example we noted that we were studying maps up to base change of the vector spaces. This is formalized in the notion of equivalent representations.

2.5 definition. A *morphism* $f\colon V \to W$ of representations of quivers is a tuple of $k$-linear maps $f_i\colon V_i \to V_j$ such that "all diagrams commute", that is, for every $\alpha\colon i \to j$, the diagram

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"'] & W_j \end{tikzcd}$$

commutes. Two morphisms $f\colon V \to W$ and $g\colon W \to X$ can be composed defining $(g \circ f)_i = g_i \circ f_i\colon V_i \to X_i$.

The definition of quivers representations and of their morphisms give rise to a $k$-linear category $\operatorname{rep}_k Q$ of representations of quivers. This automatically gives the notions of isomorphisms and of subobjects.

More precisely, we have:

- two representations $V$ and $W$ are isomorphic if there are morphisms $f_i \colon V_i \to V_j$ for every $\alpha \colon i \to j$ such that $W_\alpha = f_j \circ V_\alpha \circ f_i^{-1}$: note how this is exactly what we said before, since this states that $W$ is isomorphic to $V$ if and only if $W_\alpha$ can be obtained from $V_\alpha$ by simultaneous base changes;
- the representation $U$ is a subrepresentation of $V$ if and only if $U_i \subseteq V_i$ for every $i \in Q_0$, and $V_\alpha(U_i) \subseteq U_j$ for every $\alpha \colon i \to j \in Q_1$.

2.6 definition. The *dimension vector* of $Q$ is the vector $\underline{\dim}\, V := (\dim_k V_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$.

2.7 example. Suppose we have the quiver

$$\bullet \longrightarrow \bullet$$

then classifying all possible representations up to isomorphisms means classifying all possible matrices $m \times n$ up to invertible linear combinations of rows and of columns. We end up with only three discrete invariants: the dimensions $n$ and $m$ of $V_0$ and $V_1$ and the rank of the matrix.

2.8 example. Suppose we have the quiver

$$\bullet \circlearrowleft$$

then classifying all possible representations up to isomorphisms means classifying all possible square matrices up to conjugacy of elements in $\mathrm{GL}_n$. If $k = \overline{k}$, then we have Jordan's canonical form that gives us continuous invariants (the eigenvalues) and discrete invariants (the sizes of Jordan's blocks).

We have seen an example with only discrete invariants and one with discrete and continuous ones. We have a theorem that tells us how to discern the two cases.

2.9 theorem (Gabriel). *The classification problem for representations of the quiver $Q$ depends only on discrete invariants (that is, after fixing the dimensions of the vector spaces there are only finitely many isomorphism classes of representations) if and only if the non-oriented graph of $Q$ is a disjoint union of Dynkin diagrams of type $A_n$ ($n \geq 1$), $D_n$ ($n \geq 4$) or $E_n$ ($6 \leq n \leq 8$).*

2.10 example. Suppose you have only one vertex with two loops, that is, you want to classify two endomorphisms up to base change. This is the classic unsolved problem in linear algebra. The strategy could be to put the first endomorphism in some normal form, and then try to adjust the second without changing the good shape of the first. But this turns out to be incredibly difficult, and this is hinted by the fact that the classification problems for this quiver has $(\dim V)^2 + 1$ continuous parameters. This can be shown assuming that the first can be diagonalized with pairwise different eigenvalues; then permutations and diagonal matrices can act on this endomorphism without destroying the nice properties. But then one can prove that the second endomorphisms can be put in a form with ones on the lower diagonal, so the total number of parameters is $n + (n^2 - (n-1)) = n^2 + 1$.

*Lecture 2 (1 hour)*
*June $11^{th}$, 2012*

The example is hard also because of the high number of parameters. It turns out that this is the situation for almost all quivers.

2.11 example. The classification problem for the quiver of the previous example can be embedded into the classification problem for almost all quivers. For example, the problem for the quiver (where the number in parenthesis is the "multiplicity" of the edge, that is, the number of times it appears in $Q_1$)

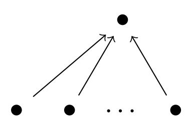
$$\bullet \xrightarrow{(2)} \bullet$$

can be solved, while

$$\bullet \xrightarrow{(3)} \bullet$$

cannot because if the two spaces have the same dimension, we can change bases so that one map is the identity; then the other two maps give precisely the same situation as before.

2.12 example. Let us consider the quiver



with $n$ source vertices. If $n = 1$, we are in the case $A_2$, so the classification problem can be solved; so it is with $n = 2$ or $n = 3$, where we have $A_3$ or $D_4$; with four we have $\widetilde{D}_4$, that can also be solved with some additional work. But in the case with five spaces below, we can decide to put the same vector space $V$ as the sources and $V^2$ as the target; we can then set the maps to be, respectively, $(\mathrm{id}, \mathrm{id})$, $(\mathrm{id}, 0)$, $(0, \mathrm{id})$, $(\mathrm{id}, \varphi)$, $(\mathrm{id}, \psi)$. Then the two maps $\varphi$ and $\psi$ gives the embedding of the quiver with one vertex and two loops.

2.13 example. Consider the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet$$

it is of type $A_n$, so by Gabriel's Theorem there are only discrete invariants. Indeed, the firsts are the dimensions of the spaces; then of course there are the ranks of the maps, but also the ranks of all the compositions. One can prove that these are all the discrete invariants one needs, even if proving this with rows and columns operations is not easy even in the case of $n = 3$.

## 2.1 *The path algebra*

2.14 theorem. *The category* $\operatorname{rep}_k Q$ *is equivalent to the category* $\operatorname{mod} kQ$ *of modules over the algebra kQ (the path algebra of Q) of finite dimension.*

2.15 definition. Let $Q$ be a quiver; $kQ$ as a vector space is spanned by paths in the quiver $Q$ (in order to have a unital algebra, there must be also the paths of length zero starting from each vertex); the multiplication of two paths is 0 if the paths cannot be composed, whereas it is the concatenation if the first path ends where the second starts. This makes $kQ$ an associative algebra with unit (given by the sum of all length zero paths).

This hints to another reason to consider only finite graph: otherwise we would have formal problems defining the unit of $kQ$.

2.16 example. If $Q$ is a vertex with a loop, then $kQ = k[T]$. If we have two loops we do not get polynomials in two variables but the free algebra in two generators, $k\langle X, Y\rangle$. If $Q$ is

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then $kQ$ is the algebra of lower triangular matrices.

How does one associate a left module over $kQ$ to a representation of $Q$? As a $k$-vector space, the module $M$ is the direct sum of all vector spaces involved, that is,

$$M := \bigoplus_{i \in Q_0} V_i\,.$$

Since the path algebra is generated by the path of length 0 or 1, we can define the action of $kQ$ on $M$ just specifying the actions of these paths. A path of length zero starting from the $i$-th component acts by selecting the $i$-th component, that is

$$(v_1, \ldots, v_k) \mapsto (0, \ldots, 0, v_i, 0, \ldots, v_i)\,;$$

instead, $\alpha\colon i \to j$ acts on $M$ mapping the component $V_i$ to $V_j$ via $V_\alpha$ and putting 0 everywhere else, that is

$$(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) \mapsto (0, \ldots, 0, \ldots, 0, \alpha(v_i), 0, \ldots, 0)\,.$$

This defines the functor on objects, and one would have to check that this definition "extends" to morphisms, and that it is an equivalence. To produce an inverse, one uses the actions of the length zero paths to split the module in the $V_i$ components, and then uses the length one paths to extract the linear maps.

2.17 example. As said before, the path algebra of

$$\bullet \longrightarrow \bullet$$

is the algebra of lower triangular matrixes, $2 \times 2$, where the element in position $(i, j)$ corresponds to the path from $i$ to $j$; if we have a representation $f\colon V \to W$, the corresponding module $M$ is $V \oplus W$, and the action of $\left(\begin{smallmatrix} a & 0 \\ b & c \end{smallmatrix}\right)$ on $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ is just the multiplication of matrices, provided that every time we use the $(2, 1)$ element we apply $f$ to $v$:

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} v \\ w \end{pmatrix} := \begin{pmatrix} a \cdot v \\ b \cdot f(v) + c \cdot w \end{pmatrix} .$$

The equivalence between $\operatorname{rep}_k Q$ and $\operatorname{mod} kQ$ is very important, because it tells us that all general results for modules over rings holds also for representations of quivers.

For example, the Jordan-Hölder Theorem, translates to the fact that all representations $V$ of $Q$ admit a filtration $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$ such that $V_{i-1}/V_i$ is *simple*, that is, it does not have subrepresentations.

Another example is the Krull-Schmidt Theorem, that states that $V$ can be written as $\bigoplus_{i=1}^n U^i$, where $U^i$ are *indecomposable* representations, that is, cannot be written as $X \oplus Y$ for $X, Y \neq 0$. Moreover, $V$ is indecomposable if and only if $\operatorname{End}(V)$ is a local ring.

Also, all the machinery of homological algebra can be used, because it is available for the module category that has enough projective.

2.18 definition. For every $i \in Q_0$, let $P_i$ be a representation of $Q$ (possibly infinite dimensional), defined letting $(P_i)_j$ be the vector space generated by paths from $i$ to $j$ (note that the vector space $(P_i)_j$ has infinite dimension when there are cycles between $i$ and $j$); define also, for $\alpha\colon j \to k$, $(P_i)_\alpha$ composing a path from $i$ to $j$ with the arrow from $j$ to $k$.

Since $kQ$, can be viewed as a module over $kQ$, it is also a representation. One can prove that it decompose as the sum of all $P_i$, and this proves that the $P_i$ are projective. In particular, we obtain $\operatorname{Hom}(P_i, V) \cong V_i$.

There is a standard projective resolution of a representation $V$. The first step is $\bigoplus_{i \in Q_0} P_i \otimes_k V_i \to V$, where the tensor product means taking as many copies of $P_i$ as the dimension of $V_i$. The sum is a finitely generated sum of projective, so it is projective. The surprising part is that the kernel of this map is automatically projective and can be described as $\bigoplus_{\alpha\colon i \to j} P_j \otimes V_i$. Summing up, the standard projective resolution of $V$ is

$$0 \to \bigoplus_{\alpha\colon i \to j} P_j \otimes V_i \to \bigoplus_{i \in Q_0} P_i \otimes_k V_i \to V \to 0 .$$

As a corollary, the Ext groups of index at least 2 are trivial in $\operatorname{rep}_k Q$ (in a similar way to what happens in the category of vector bundles on a smooth projective variety). Another corollary is that

$$\chi(M,N) = \dim \operatorname{Hom}(M,N) - \dim \operatorname{Ext}^1(M,N) = \langle \underline{\dim}\, M, \underline{\dim}\, N \rangle ,$$

where $\langle d,e \rangle := \sum_{i\in Q_0} d_i e_i - \sum_{i\to j} d_i e_j$ is a non-symmetric bilinear form, called the *Euler form* because it gives the homological Euler characteristic (the alternating sum of Ext groups).

## 3 Geometric invariant theory for quiver representation

*Lecture 3 (1 hour)*
*June 12th, 2012*

From now on we will work over an algebraically closed field $k$ of characteristic 0. We also fix a dimension vector $d \in \mathbb{N}^{Q_0}$ (since there is just one vector space of dimension $d_i$, we can think we have fixed vector spaces $V_i$ for each $i \in Q_0$).

We consider the affine $k$-space $R_d(Q) := \bigoplus_{\alpha\colon i\to j} \operatorname{Hom}_k(V_i, V_k)$. This is a space of parameters for all quiver representations of $Q$. The group $G_d := \prod_{i\in Q_0} \operatorname{GL}(V_i)$ acts on $R_d(Q)$ via $(g_i)_i \cdot (V_\alpha)_\alpha := (g_j \circ V_\alpha \circ g_i^{-1})_{\alpha\colon i\to j}$. This is a linear action of a reductive algebraic group on an affine space. Recall that we saw that if $V$ and $W$ were isomorphic representations, then we could write $W$ exactly as $g\cdot V$ for some $g \in G_d$. Hence, the orbits of $R_d(Q)$ relative to $G_d$ correspond to the isomorphism classes of representations of $Q$ with $\underline{\dim} = d$.

If we have a representation $V$, we write $[V]$ for its isomorphism class and $\mathcal{O}_V$ for the orbit of $V$ by $G_d$.

An orbit is always locally closed, so we can compute its codimension:

$$\begin{aligned}
\operatorname{codim}_{R_d(Q)} \mathcal{O}(V) &= \dim R_d(Q) - \dim \mathcal{O}_V = \\
&= \dim R_d(Q) - \dim G_d + \dim \operatorname{Stab}_{G_d}(V) = \\
&= \dim R_d(Q) - \dim G_d + \dim \operatorname{Aut}_Q(V) = \\
&= \sum_{\alpha\colon i\to j} d_i d_j - \sum_{i\in Q_0} d_i^2 + \dim \operatorname{Aut}_Q(V) = \\
&= \dim \operatorname{Aut}_Q(V) - \langle d,d \rangle .
\end{aligned}$$

This is because an element $g$ stabilizing $V$ is an automorphism of $V$ in the category of quivers representations. Note how we started from a geometric invariant (the codimension) and we could compute it using only some categorical objects: the automorphisms and the Euler form.

We also note that $\operatorname{codim}_{R_d(Q)} \mathcal{O}_V \geq 1 - \langle d,d \rangle$. We can use this fact to prove half of Gabriel's Theorem. Assume that for all dimension vectors $d$, there exist only finitely many isomorphism classes of representations of dimension vector $d$. So, for every $d$, we have finitely many orbits in the space of orbits. But orbits are locally closed, therefore they form a stratification of the space of orbits, and this imply that there must be a dense orbit. That is, for every $d$, there exists a $V \in R_d(Q)$ such that $\mathcal{O}_V \subseteq R_d(Q)$ is dense, hence of codimension 0. Thus, for every $d$, $\langle d,d \rangle \geq 1$ by the computation of the codimension of an orbit. The quadratic form $q(d) := \langle d,d \rangle$ is then positive definite, and by the theory of quadratic forms, $Q$ must be a disjoint union of Dynkin diagrams.

Of course, from the point of view of moduli of representations, the quivers that are union of Dynkin diagram are the most boring, because the associated moduli space is a union of points. In all other cases, the moduli space must be the orbit space of $R_d(Q)$ with respect to $G_d$. The only problem is that, to be sure that such a quotient exists as a geometric object, we need to use geometric invariant theory.

## 3.1 *Summary of GIT*

We will assume the setting of our moduli problem, so we will consider a linear action of a reductive group $G$ on a vector space $X$.

The basic idea is to find a variety whose functions are the invariant functions $k[X]^G$. Hilbert proved that this ring is finitely generated, so it qualifies to be the coordinate ring of some variety $X /\!/ G := \operatorname{Spec} k[X]^G$. Since $k[X]^G \subseteq k[X]$, taking the dual we have a morphism $\pi\colon X \to X/\!/G$. It satisfies several properties that are natural for quotients:

- universal property: for every morphism $\varphi\colon X \to Y$ that is $G$-invariant, there exists exactly one way to split this morphism through $\pi$;
- surjectivity;
- each fiber of $\pi$ contains a unique closed $G$-orbit (if one orbit is in the closure of another, we cannot expect that a morphism is able to distinguish between them);
- closed $k$-points of $X/\!/G$ corresponds to closed $k$-orbits of $X$.

This was a summary of pre-Mumford GIT. Now, what we would like is for $\pi$ to actually separate all orbits. As said before, the only way to do this is to restrict our space $X$ to $X^{\mathrm{st}}$, the *stable* locus, that is, the locus of points $x \in X$ such that $\mathrm{Stab}_G(x)$ is finite. If we restrict to $X^{\mathrm{st}}$, then the map $\pi$ actually has one orbit per fiber.

3.1 example. Suppose that $\mathbb{G}_m$ acts on $\mathbb{A}^2$ as $t \cdot (x, y) := (tx, ty)$. The orbits are all the rays (not closed) and the origin (closed), so $\mathbb{A}^2 /\!/ \mathbb{G}_m = \{0\}$. Instead, the aim is to have $\mathbb{A}^2 /\!/ \mathbb{G}_m = \mathbb{P}^1$.

*Lecture 4 (1 hour)*
*June $13^{th}$, 2012*

Recall that to do Mumford-style GIT we need also a line bundle on the space $X$ with a $G$-linearization. This blows down to the choice of a character $\chi\colon G \to \mathbb{G}_m$, and we can define the set of $\chi$-semi-invariant functions as

$$k[X]^{G,\chi} := \{f\colon X \to k \mid \forall g \in G, \forall x \in X, f(gx) = \chi(g) f(x)\}.$$

The problem is that if we multiply two $\chi$-semi-invariant functions we get a $\chi^2$-semi-invariant function, and so $k[X]^{G,\chi}$ is not a ring. What we can do is to take all $\chi^n$-semi-invariant functions, for every $n$, and construct a graded ring $k[X]^G_\chi := \bigoplus_{n \geq 0} k[X]^{G,\chi^n}$. Our hope is that the Proj of this graded ring is a sensible quotient. We need some other definition: $X^{\chi-\mathrm{sst}}$ is the locus of points $x \in X$ such that $f(x) \neq 0$ for some $f \in k[X]^G_\chi$ of positive degree; inside $X^{\chi-\mathrm{sst}}$ we define $X^{\chi-\mathrm{st}}$ as the locus of points with the additional condition of having a finite stabilizer. Both the semi-stable locus and the stable locus are open.

From now on, we will consider not $X$ but $X^{\chi-\mathrm{sst}}$, so if we say "closed orbit" it will mean "closed in $X^{\chi-\mathrm{sst}}$".

The situation is the following

$$\begin{tikzcd}
X^{\chi-\mathrm{st}} \arrow[r, hook, "\text{open}"] \arrow[d, "\pi"] & X^{\chi-\mathrm{sst}} \arrow[r, hook, "\text{open}"] \arrow[d, "\pi"] & X \arrow[d, "\pi"] \\
X^{\chi-\mathrm{st}}/G \arrow[r, hook, "\text{open}"] \arrow[d, equal] & X^{\chi-\mathrm{sst}}/\!\!/G \arrow[r, "\text{projective}"] \arrow[d, equal] & X/\!\!/G \arrow[d, equal] \\
\operatorname{Proj} k[X]^G_\chi & \operatorname{Proj} k[X]^G_\chi & \operatorname{Spec} k[X]^G
\end{tikzcd}$$

where the morphism $X^{\chi-\mathrm{sst}}/\!\!/G \to X/\!\!/G$ is projective because the latter's ring is the degree 0 of the former's ring. If the action of $G$ on the stable locus is not only with finite stabilizers but also free, then $X^{\chi-\mathrm{st}}/\!\!/G$ is smooth and the map $\pi$ over it is a $G$-principal bundle.

3.2 example. In the case of $\mathbb{A}^2$ with the action of $\mathbb{G}_m$ by dilations, we can choose the trivial character $\chi = \mathrm{id}$, and we obtain $k[\mathbb{A}^2]^{\mathbb{G}_m}_\chi = k[X, Y]$, where $X$ and $Y$ are $\chi$-semi-invariant functions of degree 1. It's Proj is of course $\mathbb{P}^1$. We can check also that the semi-stable locus is $\mathbb{A}^2 \setminus \{0\}$.

3.3 example. Look at the actions of $\mathbb{G}_m$ on $\mathbb{A}^2$ by $t \cdot (x, y) := (tx, ty^{-1})$, where the generic orbit is an hyperbola, and we have three special orbit: the axes and the origin. Choosing the identity character, one sees that the stable locus consists of all the hyperbolas and the punctured $x$ axis; choosing instead the character $t \mapsto t^{-1}$, the semi-stable locus comprises all hyperbolas and the punctured $y$-axis. This shows that the semi-stable locus is not intrinsic: we have to make a choice, and this choice is summed up in the choice of the character.

3.4 example. We can define grassmannians using GIT by acting with $\mathrm{GL}_k$ on $M_{n\times k}$ (with $k \leq n$) and using the determinant as the character. The quotient is the grassmannian $G(k, n)$.

### 3.2 *Application to the action of $G_d$ on $R_d(Q)$*

Recall that we associated to the class of a representation $[V]$ the orbit $\mathcal{O}_V$. The problem is that if we apply GIT directly, we obtain stabilizers of positive dimension, hence the stable locus would be empty by definition. But there is a one dimensional family of elements of $G_d$ acting trivially on $R_d(Q)$: the elements that rescale the bases of all vector spaces by the same amount. Therefore we can consider $\mathbb{P}G_d$ as $G_d$ modulo the scalars. Since $\mathrm{Stab}_{G_d}(V) = \mathrm{Aut}(V)$, we have that $\mathrm{Stab}_{\mathbb{P}G_d}(V) = \mathrm{Aut}(V)/\mathbb{G}_m$.

Now, the only decent characters $\chi\colon \mathbb{P}G_d \to \mathbb{G}_m$ are the ones associating to the matrices $g_i$ the product of their determinants to some power $\vartheta_i$: $\chi((g_i)_i) = \prod_i \det(g_i)^{\vartheta_i}$. But, to make this well defined for $\mathbb{P}G_d$, there is the additional condition that $\sum d_i\vartheta_i = 0$. We will write $\chi_\vartheta$ for the character $\chi$.

Applying the GIT theory we get

$$\begin{array}{ccccc} M_d^{\vartheta-\mathrm{st}}(Q) & \xhookrightarrow{\text{open}} & M_d^{\vartheta-\mathrm{sst}}(Q) & \xrightarrow{\text{projective}} & M_d^{\mathrm{ssimp}}(Q) \\ \| & & \| & & \| \\ R_d(Q)^{\chi_\vartheta-\mathrm{st}}/\mathbb{P}G_d & & R_d(Q)^{\chi_\vartheta-\mathrm{sst}}/\!/\mathbb{P}G_d & & R_d(Q)/\!/\mathbb{P}G_d \end{array}$$

As before, the last term is affine, and the first is smooth because the stabilizer is $\mathrm{Aut}(V)/\mathbb{G}_m$, that is a connected, zero-dimensional group inside $\mathrm{End}(V)/\mathbb{G}_m$: stabilizers are trivial.

We used the index "semi-simple" for the affine quotient. This is because one could prove that the closed orbits correspond to representations $V$ that can be decomposed in a sum of irreducible representations, that is, $V$ is semi-simple. That is, $M_d^{\mathrm{ssimp}}(Q)$ parametrizes semi-simple representation of $Q$ (with $\underline{\dim} = d$).

If $Q$ has no oriented cycles, the only irreducible representations are the $S_i$.[1] This implies that the only closed orbit in $R_d(Q)$ is $\{0\}$ and $M_d^{\mathrm{ssimp}}(Q) = \{\mathrm{pt}\}$: it is a situation similar to the one of $\mathbb{A}^2$, where we had only one closed orbit (the origin), that "masked" all other interesting non-closed orbit.

In the case $Q$ has oriented cycles, a theorem of Le Bruyn-Procesi states that $k[R_d(Q)]^{G_d}$ is generated by the traces of the compositions of the maps along the oriented cycles. More precisely, by functions of the form $V \mapsto \mathrm{Tr}(V_{\alpha_1} \circ V_{\alpha_n} \circ \cdots \circ V_{\alpha_2} \circ V_{\alpha_1})$ for a cycle $\alpha_1 \circ \alpha_n \circ \cdots \circ \alpha_1$. The problem is that the theorem is non-constructive, hence we don't know how many of such trace functions we need to take to generate the whole ring (and which are the relations amongst the generators).

3.5 example. Consider the quiver $Q$ with one vertex and two loops, then $M_d^{\mathrm{ssimp}}(Q) = \mathbb{A}^5$ when $d = 2$, and the generators of the ring are the traces of $A$, $B$, $AB$, $A^2$, $B^2$. If $d = 3$, we have 10 traces with one relation. If we have $d = 3$ with three loops, one needs to take 48 traces and there are 365 relations.

## 4 Slope stability

*Lecture 5 (1 hour)*
*June $14^{th}$, 2012*

We have an interpretation for the quotient $M_d^{\mathrm{ssimp}}(Q)$ (that is, it parametrizes semi-simple representations), but we still have to discover interpretations for the two spaces $M_d^{\vartheta-\mathrm{sst}}(Q)$ and $M_d^{\vartheta-\mathrm{st}}(Q)$. There is a nice interpretations analogous to the one found in moduli spaces of vector spaces, using slope stability.

We define the rank of a dimension vector as the sum of all dimensions of the vector spaces, and we choose an arbitrary degree function $\deg\colon \mathbb{Z}^{Q_0} \to \mathbb{Q}$, which is simply a linear function. In the setting of vector bundle on curves we have a natural degree function, here we do not.

Given a degree function, we can define the slope function as

$$\mu\colon \mathbb{N}^{Q_0} \setminus \{0\} \to \mathbb{Q}$$

[1] Maybe the $P_i$?

by $\mu(d) = \deg d / \operatorname{rk} d$. We have to exclude 0 because its rank is zero.

4.1 definition. A quiver representation $V$ is called *semi-stable* if $\mu(\underline{\dim} U) \leq \mu(\underline{\dim} V)$ for every $U \subsetneq V$ nonzero. It is called *stable* if the same holds with a strict inequality. $V$ is called *poly-stable* if $V$ is a direct sum of stable representations with the same slope.

Inside $\operatorname{rep}_k Q$ one can find the subcategory of semi-stable representations of slope $\mu$, and it is an abelian subcategory, meaning that, for example, kernels and cokernels are inside the subcategory. Moreover, inside it, the stable objects are precisely the irreducible one with respect to the subcategory.

Given a dimension vector $d$, we can define $\vartheta_i := \deg(d) - \deg(e_i) \cdot \operatorname{rk}(d)$ (where $e_i$ is the dimension vector $(0, \ldots, 0, 1, 0, \ldots, 0)$). Using this $\vartheta$, we have $\sum \vartheta_i d_i = 0$ as requested to let the action descend to $\mathbb{P}G_d$. This is a natural choice for any given degree function deg and dimension vector $d$.

4.2 theorem (King). *A representation $V$ is in $R_d^{\chi_\vartheta-\mathrm{sst}}(Q)$ if and only if $V$ is semi-stable; $V \in R_d^{\chi_\vartheta-\mathrm{st}}(Q)$ if and only if $V$ is stable; $\mathcal{O}_V \subseteq R_d^{\chi_\vartheta-\mathrm{sst}}(Q)$ is closed if and only if $V$ is poly-stable.*

Note that on the right sides we used the notion of stability, that is relative to a degree function. On the left, this choice is hidden in the $\vartheta$, that is constructed to adapt to the chosen degree function. Therefore, from now on we drop $\vartheta$ from the notation.

4.3 corollary. *The space $M_d^{\mathrm{sst}}(Q)$ parametrizes isomorphism classes of poly-stable representations with $\underline{\dim} = d$; the (smooth) space $M_d^{\mathrm{st}}(Q)$ parametrizes isomorphism classes of stable representations with $\underline{\dim} = d$.*

We have no direct way to say that a space $M_d^{\mathrm{st}}(Q)$ is nonempty: there are some criterions, but they are highly recursive criterions, hence not very effective. Let us assume to have $M_d^{\mathrm{st}}(Q)$ nonempty, then we could ask what is its dimension. Of course, $\dim M_d^{\mathrm{st}}(Q) = \dim R_d(Q) - \dim \mathbb{P}G_d$, that we already seen being equal to $1 - \langle d, d \rangle$. A necessary condition for the moduli space to be nonempty, is then that $\langle d, d \rangle \leq 1$. This in particular shows that for Dynkin quivers, being the Euler form positive definite, the dimension is 0 (or the moduli space is empty).

What happens when we change the degree function? For example, let $Q$ be the quiver

$$\bullet \xrightarrow{(m)} \bullet$$

and let the dimension vector be $d := (d_1, d_2)$. The degree is a linear function, hence of the form $\deg(d) := a_1 d_1 + a_2 d_2$ with $a_i \in \mathbb{Z}$. If we plot the coefficients $a_i$ in a $\mathbb{Z}^2$, then we can identify a locus, $(a_1 = a_2)$ on which $R_d^{\chi_\vartheta-\mathrm{sst}}(Q) = \{0\}$, hence $M_d^{\mathrm{sst}} = \{\mathrm{pt}\}$. If we choose a degree function above the diagonal, namely with $(a_2 > a_1)$, then $R_d^{\chi_\vartheta-\mathrm{sst}}(Q) = \varnothing$, except for $d_1 d_2 = 0$, for which the moduli space is anyway a single point. In the other directions instead we have more interesting results. Here a representation with maps $f_i$ is (semi-)stable if and only if for every nonzero $U \subsetneq V$ we have $\dim \sum_{i=1}^{m} f_i(U) \geq \dim U \cdot \frac{\dim W}{\dim V}$. This happens when the linear maps $f_i$ are in some sense "orthogonal" to each other, in particular it is a genericity condition that is open. This indeed is the so called Kronecker moduli space, studied by Drezet. But even in this case it is not easy to decide whether the stable locus is nonempty.

Note that for this quiver, a subrepresentation is a choice of vector spaces $U_V \subseteq V$ and $U_W \subseteq W$ such that $f_i|_{U_V} \subseteq U_W$ but $U_W$ must contain also the sum of all images: $U_W \supseteq \sum f_i(U_V)$.

4.4 example. We look at moduli spaces of small dimension. For example, we choose the quiver

$$\begin{tikzcd} & & 2 & \\ 1 \arrow[urr] & 1 \arrow[ur] & 1 \arrow[u] & 1 \arrow[ul] \end{tikzcd}$$

with the dimension vector specified in the diagram. A representation of this is a configuration of 4 vectors in a 2-dimensional vector space, or passing to the projective spaces, 4 points in $\mathbb{P}^1$. The moduli space is indeed parametrized by the cross-ratio: for example, with $\deg(s_1, \ldots, s_4, t) = -e$, $M_d^{\mathrm{sst}}(Q) \cong \mathbb{P}^1$, and $M_d^{\mathrm{st}}(Q) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

4.5 example. In the case of five maps with a common target, the stable or semi-stable moduli space is $\mathbb{P}^2$ blown up in four points. If the quiver and dimension vector is

$$\begin{tikzcd} & 1 & \\ 1 \arrow[ur] \arrow[r] \arrow[dr] & 1 & 1 \arrow[ul] \arrow[l] \arrow[dl] \\ & 1 & \end{tikzcd}$$

then the moduli spaces (both stable and semi-stable) are $\mathbb{P}^2$ blown up in three points.

4.6 example. The quiver and dimension vector $1 \xrightarrow{(m)} k$ has moduli spaces $G(m, k)$; $2 \xrightarrow{(3)} 3$ admits a $\mathbb{G}_m$ action with 13 fixed points, and for both the stable and semi-stable moduli spaces coincide.

4.7 conjecture. *If $M_d^{\mathrm{st}}(Q) = M_d^{\mathrm{sst}}(Q)$, then it admits an affine pairing.*

## 5 Cohomology of quiver moduli

*Lecture 6 (1 hour)*
*June $14^{th}$, 2012*

We will assume that $Q$ has no oriented cycles, and that $d$ is deg-coprime, that is, for every nonzero $e < d$, $\mu(e) \neq \mu(d)$ (in other words, $\gcd(d_i) = 1$ and degree is "generic", avoiding finitely many hyperplanes). For example, in the case $\bullet \xrightarrow{(m)} \bullet$, the only reasonable degree function is $\deg(d_1, d_2) = d_1$, and we also require that $\gcd(d_1, d_2) = 1$.

These requirements implies that $M_d(Q) := M_d^{\mathrm{st}}(Q) = M_d^{\mathrm{sst}}(Q)$, because the equality in the semi-stability condition cannot be reached under these conditions. In particular, $M_d(Q)$ is smooth because the stable locus is smooth, and is projective because the semi-stable locus is projective.

### 5.1 *Harder-Narasimhan stratification of the unstable locus*

5.1 lemma (Harder-Narasimhan). *For all representations $V$, there exists a unique filtration $0 = V^0 \subsetneq V^1 \subsetneq \cdots \subsetneq V^s = V$ such that all quotients $V^i/V^{i-1}$ are semi-stable and the slopes are decreasing, that is, $\mu(V^1/V^0) > \mu(V^2/V^1) > \cdots > \mu(V^s/V^{s-1})$.*

The surprising fact about this lemma is that the filtration respecting these properties is unique. Moreover, with some care it can be done in such a way that it also respect functoriality (that is, in such a way that it is preserved by morphisms). Thanks to the uniqueness we can define the following.

5.2 definition. The *Harder-Narasimhan type* of a representation $V$ is the list $(\underline{\dim}\, V^1/V^0, \ldots, \underline{\dim}\, V^s/V^{s-1}) \in (\mathbb{N}^{Q_0})^s$.

Obviously, if $(d^1, \ldots, d^s)$ is a HN type, then $\sum d^i = d$ (this implies that after fixing the dimension vector, there are only finitely many HN types for that dimension vector). Also, by definition, $\mu(d^1) > \cdots > \mu(d^s)$. Moreover, if $V$ is semistable, then the HN filtration is $0 \subsetneq V$. The HN type is therefore $(d)$, and this condition is equivalent to semi-stability.

Because of the finiteness condition, we can stratify the space $R_d(Q)$ by the HN type:

$$R_d(Q) = \bigsqcup_{\substack{d^\star = (d^1, \ldots, d^s), \\ \sum d^i = d}} \mathcal{S}_{d^\star} = \bigsqcup_{\substack{d^\star = (d^1, \ldots, d^s), \\ \sum d^i = d}} \{V \in R_d(Q) \mid \text{HN type of } V \text{ is } d^\star\} .$$

It is a finite stratification, and $\mathcal{S}_{(d)}$ corresponds to the $\chi_\vartheta$-semi-stable locus, that is open in $R_d(Q)$. Moreover, since $R_d(Q)$ is a vector space, we know that it is the only open stratum.

It would be great if the stratum $\mathcal{S}_{(d^1, \ldots, d^s)}$ was related to $\prod_{k=1}^s R_{d^k}^{\chi_\vartheta - \mathrm{sst}}(Q)$, that is, the "boundary" strata were constructed starting from moduli of smaller representations. What happens is slightly more complicated, that is we have

$$\mathcal{S}_{(d^1, \ldots, d^s)} \cong G_d \times_{P_{d^\star}} B ,$$

where $B$ is a vector bundle over $\prod_{k=1}^s R_{d^k}^{\chi_\vartheta - \mathrm{sst}}(Q)$. The group $P_{d^\star}$ is a parabolic subgroup in $G_d$ with Levi factor $\prod_{k=1}^s G_{d^k}$ (that is, we are taking upper triangular block matrices with diagonal blocks equal to $G_{d^k}$, inside $G_d$ that is a product of GL groups).

Note that we are not stratifying the moduli space directly, we are stratifying the big parameter space $R_d(Q)$. This does not give directly ways to decompose the moduli space in a useful way to compute invariants.

Consider $K_0(\mathrm{Var}\,/k)$ that is the set of all isomorphism classes of quasi-projective varieties $X$ up to relations of the form $[X] = [A] + [U]$ whenever $A \subseteq X$ is closed and $U = X \setminus A$. We put an operation on this set, that is $[X] \cdot [Y] := [X \times Y]$, that makes it into a ring. We can define operators on this ring together with the formal sum. To keep things simple, we consider $k = \mathbb{C}$; then for example, we can define $\chi\colon K_0(\mathrm{Var}\,/\mathbb{C}) \to \mathbb{Q}$ sending $[X]$ to $\chi_c(X)$, the Euler characteristic with compact support. This is known to respect the relation for which we quotiented, so it is well defined. Another operator we can define on $K_0(\mathrm{Var}\,/\mathbb{C})$ is the virtual Poincaré polynomial, that is a ring homomorphism $K_0(\mathrm{Var}\,/\mathbb{C}) \to Q[q]$ sending $[X]$ with $X$ smooth and projective to $\sum_{i\geq 0} \mathrm{h}^i(X,\mathbb{Q})q^i$. It can be proved that there is only one ring homomorphism with this behaiour for smooth projective varieties. Another possibility is to count points over finite fields, that respect the relation and is multiplicative with respect to the cartesian product.

What is the class of $R_d(Q)$ in $K_0(\mathrm{Var}\,/k)$? Since it is an affine space, it is just $[\mathbb{L}]^{\dim R_d(Q)}$, the class of a line with the appropriate exponent. But we have the HN filtration on $R_d(Q)$, so

$$[R_d(Q)] = [R_d^{\chi_\vartheta-\mathrm{sst}}(Q)] + \sum_{d^\star \text{ proper HN type}} [\mathcal{S}_{d^\star}]\,,$$

and we can express the classes in the sum using the fornula saw before:

$$[R_d(Q)] = [R_d^{\chi_\vartheta-\mathrm{sst}}(Q)] + \sum_{d^\star \text{ proper HN type}} \frac{[G_d]\cdot[\prod_{k=1}^s R_{d^k}^{\chi_\vartheta-\mathrm{sst}}(Q)]\cdot[\mathbb{L}]^{\dim B}}{[P_{d^\star}]}\,.$$

In the end we get the motivic HN recursion:

$$\frac{[R_d^{\mathrm{sst}}(Q)]}{[G_d]} = \frac{[R_d(Q)]}{[G_d]} - \sum_{d^\star}[\mathbb{L}]^{\dim B}\prod_{k=1}^{s}\frac{[R_d^{\mathrm{sst}(Q)}]}{[G_{d^k}]}\,.$$

This is a completely formal relation that we can use to compute the motive of the moduli space $M_d(Q)$, in terms of things that are either motives of smaller dimensional moduli of representations, or known such as affine spaces.

If $d$ is deg-coprime, then, for example, $[R_d^{\mathrm{sst}}(Q)]/[G_d]$ and $[M_d(Q)]/[\mathbb{G}_m]$ have the same virtual Poincaré polynomial.

## 6 Non-commutative algebraic geometry

*Lecture 7 (1 hour)*
*June 15th, 2012*

There are two approaches when you decide to do non-commutative algebraic geometry: the small scale approach is when you actually take non-commutative algebras, but such that they are of finite rank over their center (so that you can hope that many properties extends); the large scale approach is when you instead take “very” non-commutative algebras, for example the free algebra $A := k\langle x_1, \dots, x_n\rangle$. Here we will follow the second approach.

6.1 example. We define $\mathbb{A}^m := \operatorname{Specm} k[x_1, \dots, x_m]$, where Specm is, set-theoretically, the set of maximal, two-sided ideals. The non-commutative analogous is $N\mathbb{A}^m := \operatorname{Specm} k\langle x_1, \dots, x_m\rangle$. Inside it, we can find the cofinite part, that is $\operatorname{Specm} \operatorname{cofin} k\langle x_1, \dots, x_m\rangle = \{\mathfrak{m} \mid \dim_k k\langle x_1, \dots, x_m\rangle/\mathfrak{m} < \infty\}$. of course in the commutative case this is equivalent to Specm. We attach to these spaces a topology that is constructed in the same way as the Zariski topology.

The nice thing is that we can construct the cofinite part using quivers. The following is a reinterpretation of a theorem initially stated in a different language.

6.2 theorem (Artin, 1969). *We have*

$$\operatorname{Specm} \operatorname{cofin} k\langle x_1, \dots, x_m\rangle \cong \bigcup_{d\geq 1} M_d^{\mathrm{simp}}(m \textit{ loops}) .$$

In the theorem, $M_d^{\mathrm{simp}}(Q)$ is the open part of $M_d^{\mathrm{ssimp}}(Q)$ corresponding to $R_d^{\mathrm{st}}(Q)$. For example, in the case of a single vertex with $m$ loops, we have that the simple representations are the ones where there is not a common invariant subspace for all the linear maps. In particular, for $m = 1$ there are no simple representations.

The correspondence in the theorem sends the class of $V = (\varphi_1, \dots, \varphi_m)$ to its annihilator, $\operatorname{Ann}(V) = \{P \in k\langle x_1, \dots, x_m\rangle \mid P(\varphi_1, \dots, \varphi_m) = 0\}$. We can now reformulate the theorem of Le Bruyn-Procesi and see the generators inside the non-commutative ring: we can see $\operatorname{Specm} \operatorname{cofin} k\langle x_1, \dots, x_m\rangle$ inside $\operatorname{Tr} \mathbb{C}\langle x_1, \dots, x_m\rangle$, the set of trace-like functions, that is of linear functions $t\colon \mathbb{C}\langle x_1, \dots, x_m\rangle \to \mathbb{C}$ such that $t(PQ) = t(QP)$. This embedding is realized sending the representation $V$ to the map $P \mapsto \operatorname{Tr} P(\varphi_1, \dots, \varphi_m)$.

Differently from the usual algebraic geometry where the affine space is one of the simplest object, in the non-commutative case the affine space is already very wild. For example, a notorious hard problem in quiver representations is the rationality of $M_d^{\mathrm{ssimp}}(Q)$: this problem can be reduced to the case where $Q$ consists of one vertex with $m$ loops, and indeed it is not even known if the non-commutative affine space is rational.

6.3 example. The scheme $\operatorname{Hilb}_d(\mathbb{A}^m)$ is the space parametrizing zero dimensional subschemes of $\mathbb{A}^m$ of length $d$. These are in turn in a correspondence with ideals $I \subseteq k[x_1, \dots, x_m]$ such that $\dim_k k[x_1, \dots, x_m]/I = d$. In the non-commutative case, the analogous is $N\operatorname{Hilb}_d^{(m)}$, the space of left ideals $I$ in $k\langle x_1, \dots, x_m\rangle$ such that $\dim_k k\langle x_1, \dots, x_m\rangle/I = d$. This quotient is no longer a ring but only a vector space (because we asked for left ideals), but we actually just care about the dimensions. In reality, the left ideal are choosen because they are easier to figure out with respect to the more natural definition with two-sided ideals.

We define a quiver $Q$ with a map between two vector spaces of dimension 1 and $d$, with the target having also $m$ loops. The dimension vector is $\underline{d} = (1, d)$ and we choose the degree to be $\deg(a, b) = a$. Choosing a representation is the same as choosing a vector $v$ in the second space plus $m$ linear maps. It is (semi-)stable if and only if $v$ is a cyclic vector with respect to the other maps, that is, applying all possible polynomials of $\varphi_1, \ldots, \varphi_m$ to $v$, we can arrive to any vector.

6.4 lemma. *The moduli space $M^{\mathrm{st}}_{\underline{d}}(Q)$ is isomorphic to $N\operatorname{Hilb}^{(m)}_d$. We associate to a point $[(v, \varphi_1, \ldots, \varphi_m)]$ the annihilator $\{P \mid P(\varphi_1, \ldots, \varphi_m)(v) = 0\}$. In the other direction, we associate to a left ideal $I$ the point $[(\overline{1} \in A/I, \overline{x_1}, \ldots, \overline{x_m})]$, where $x_i$ are the generators of the ideal.*

The advantage in the first case was that we got natural coordinates in $M^{\mathrm{simp}}_d(Q)$ using the embedding in the trace-like function using the Le Bruyn-Procesi Theorem. In this case, the advantage is that we can apply HN recursion to get a recursive formula for the Betti numbers of $N\operatorname{Hilb}^{(m)}_d$. We can construct a two variables generating function putting together all Betti numbers of $N\operatorname{Hilb}^{(m)}_d$:

$$F(q, t) = \sum_{d \geq 0} q^{(m-1)\binom{d}{2}} \sum_i \dim \mathrm{H}^i(N\operatorname{Hilb}^{(m)}_d, \mathbb{Q}) q^{-i/2} t^d \,.$$

The twist $q^{-i/2}$ comes from the fact that these spaces have no odd cohomology.

6.5 theorem. *The function $F(q, t)$ is determined by the following algebraic $q$-different functional equation:*

$$F(t, q) = \frac{1}{1 - F(q, qt)F(q, q^2 t) \cdots F(q, q^{m-1} t)} \,.$$

Using this equation one can prove things like a formula for the Euler characteristic:

$$e(N\operatorname{Hilb}^{(m)}_d) = \frac{1}{(m-1)d + 1} \binom{md}{d} \,.$$

Let us consider $\sum_{n \geq 0} e(\operatorname{Hilb}_n(\mathbb{A}^i)) t^n$; when $i \in \{2, 3\}$, we can rewrite this as $\prod_i \geq 1 (1 - t^i)^{-j}$ where $j = 1$ for $i = 2$ and $j = i$ for $i = 3$. We can do the same for $N\operatorname{Hilb}^{(m)}_d$, where $F(1, t) = \prod_{i \geq 1} (1 - t^i)^{-i\,\mathrm{DT}^{(m)}_i}$, where $\mathrm{DT}^{(m)}_i$ is the Donaldson-Thomas invariant of the Hilbert scheme, for which it exists a formula using the formula for the Euler characteristic.