Cohomological and birational aspects of the moduli space of curves

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1 Introduction

Lecture 1 (1 hour) June 11th, 2012

The story starts in 1857, with the famous paper of Riemann, in which he defined a space parametrizing the isomorphisms classes of Riemann surfaces of fixed genus. He computed correctly the dimension of this space, $\mathcal{M}_g$, that is $3g-3$ (when $g \geq 2$).

In 1890, Clebsch and Hurwitz proved that $\mathcal{M}_g$ is connected using $\mathcal{H}_{g,k}$, the space of $k:1$ covers of $\mathbb{P}^1$ of genus g; this space has two projections, to $\mathcal{M}_g$ and to $\mathbb{P}^b = \operatorname{Sym}^b \mathbb{P}^1$ (since each cover has $b = 2g+2k-2$ simple branch points). If we fix b points on $\mathbb{P}^1$, then to give a cover is equivalent to give the monodromy, that is a morphism $\varrho_f \colon \pi_1(\mathbb{P}^1 \setminus \{p_1, \dots, p_b\} \to S_k$, with the basic loops mapping to transpositions (to force the branch points to be simple). Then Clebsch and Hurwitz proved that the covering space is connected. A modern treatment of this can be found in the 1969 paper by Fulton.

In 1943, Teichmüller constructed $\mathcal{M}_g$ in a completely different way, starting from what it is now called the Teichmüller space and taking a quotient by the mapping class group. It is constructed as an analytic variety.

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Despite all the results proved on $\mathcal{M}_g$, only in 1965 Mumford proved that it actually exists (as an algebraic variety). Indeed, Riemann only defined it as a set, which is not really useful by itself.

Having $\mathcal{M}_g$ defined as a variety gives some interesting properties. A minimalistic one is that if we have a flat family $f\colon X \to T$ of smooth curves of genus g, then there should be an associated algebraic function $m(f)\colon T \to \mathcal{M}_g$ such that $m(f)(t) = [X_t]$ for each $t \in T$.

More sofisticated properties require discussing the difference between a coarse and a fine moduli space. Consider the (contravariant) moduli functor $F_{M_g}\colon \mathfrak{Sch} \to \mathfrak{Sets}$ that sends a scheme to the set of flat families over T of curves of genus g. We say that a space $\mathcal{M}_g$ is a fine moduli space of curves if F_{M_g} is equivalent to the functor $\mathrm{Hom}(\bullet, \mathcal{M}_g)$.

Suppose we have a fine moduli space $\mathcal{M}_g$; then we have a special element in $\mathrm{Hom}(\bullet, \mathcal{M}_g)$, namely $\mathrm{id}_{\mathcal{M}_g}$; this would be associated to a universal family of curves $\mathcal{C} \to \mathcal{M}_g$ that would induce every flat family of curves. But this is not possible in general, because not every isotrivial family (with all fibers isomorphic) is trivial. This has to do with the fact that curves have non-trivial automorphisms.

1.1 example. Take a curve of genus at least 2 with an automorphism (for example, a hyperelliptic curve); take a non-trivial subgroup G of $\mathrm{Aut}(C)$; then G acts on all elliptic curves by translations. The quotient $C \times E/G$ is a fibration to the elliptic curve $E/G = D$, whose fibers are all isomorphic to C, but it is not trivial.

There are two ways now that one can walk through: the first started with Mumford in the sixties, and consists of considering stacks instead of schemes, where the functor F_{M_g} is representable. The other is to consider just the coarse moduli space $\mathcal{M}_g$. It is defined as a space that admits a natural transformation $F_{M_g} \to \mathrm{Hom}(\bullet, \mathcal{M}_g)$ (this is the minimalistic property we asked before); another property we want is that the points of this space are in a bijection with the actual isomorphism classes of curves, that is, $F_{M_g}(k) = \mathrm{Hom}(k, \mathcal{M}_g)$ for every field k algebraically closed; finally, we require a universal property: every natural transformation $F_{M_g} \to \mathrm{Hom}(\bullet, N)$, where N is a scheme, satisfying the second property, factors through the natural transformation to $\mathrm{Hom}(\bullet, \mathcal{M}_g)$.

Mumford proved that this coarse moduli space exists (as a quasi-projective scheme) using geometric invariant theory (nowadays this can be proved more economically using Kuranishi families). After Mumford's proof, all the theorems on the structure of $\mathcal{M}_g$ could be applied to the space $\mathcal{M}_g$ constructed by him.

2 Compactification

Consider the family $y^2 = x^3 + tx^2$: it has as fiber a smooth elliptic curve for each t, but for $t = 0$ it has a cusps. One would like to fill this point in a consistent way. This has been done in 1969 by Deligne and Mumford using stable curves. A stable curve is a nodal curve with finite automorphism group, or such that every rational component intersect the rest of the curve in at least three points. They proved that adding these curves is enough to produce a compact space, $\overline{\mathcal{M}}_g$.

Lecture 2 (1 hour)
June 12th, 2012

A good way to visualize stable curves is by using their dual graph, with a vertex for each irreducible component and an edge for every node. Each graph correspond to a topological stratum of $\overline{\mathcal{M}}_g$, and these strata are parametrized by moduli spaces of curves of lower genera. The dual graph language comes really handy when you would like to compute the intersection of two strata and degenerations of curves. Indeed, a graph corresponds to a stratum of codimension equal to the number of edges in the graph, and if a graph H is obtained from G collapsing an edge, we say that G is a specialization of H, and we know that the stratum of G is in the closure of the stratum of H.

Why is $\overline{\mathcal{M}}_g$ complete? The answer lies in the stable reduction theorem, that states that given a one dimensional family of curves (over a DVR, or a disk), with all fibers but the one over 0 smooth, we can replace the fiber over 0 with a stable curve (maybe after doing a finite base change). This is equivalent to the statement that given a map from the disk without 0 to $\overline{\mathcal{M}}_g$, we can always extend the map. The proof uses the standard theory of resolution of singularity for a surface: after blowing up we can assume that the total space of the family (which is a surface) is smooth, and that the central fiber has normal crossing singularities, but it could be non-reduced. The finite base change is needed exactly to make the central fiber reduced.

2.1 example. Suppose we have the family $y^2 = x^3 + t$; we have a cusp point in the central fiber, so we blow up this point to obtain a smooth curve $\widetilde{C}$ tangent to the exceptional divisor E_1, which is not good enough (we want normal crossing). The fiber is $2E_1 + \widetilde{C}$. Blowing up again, we will have three curves, E_1, E_2 and $\widetilde{C}$, all converging to a point, and the fiber is $\widetilde{C} + 2E_1 + 3E_2$. Blowing up again we obtain a nodal curve, E_3 intersecting each of the other three curves in a point; the fiber is then $\widetilde{C} + 2E_1 + 3E_2 + 6E_3$. Now we have to perform a base change, the smallest one is the lcm of the coefficients. In this situation is easier to perform two base changes, of prime order (that is easier). Geometrically, if the central fiber is $\sum a_i D_i$, a base change of finite order r has the effect of taking a branch cover over the divisor $\sum (a_i \mod r) D_i$. So, taking a base change of order 2 is like a taking a branched cover over $\widetilde{C} + E_2$; these two divisors survive with the same multiplicity; E_1 will be replaced by two components of multiplicity 1, and E_3 survive (since the quotient of a rational curve is rational) with multiplicity $3 = 6/2$. Taking the base change of order 3 is now like taking a branch cover over $\widetilde{C} + E_1' + E_1''$, and E_2 get replaced by three copies; E_3 is substituted by a cover of order 3 of E_3 ramified on 3 points, that is, an elliptic curve. We can now blow down the -1 cirves and the only components remaining are $\widetilde{C} + \widetilde{E}_3$. It is interesting to note that $\widetilde{E}_3$ is not a random elliptic curve, that it is the Fermat cubic ($j = 0$).

2.2 example. A similar example is when we identify two points p and q on a curve C, and we let $q \to p$. The family we can choose is $C \times C$, identifying two sections, one constant and one diagonal. Here we just need to blow up once the cusp, and perform no base changes. The additional component will be the elliptic curve with $j = \infty$ with two points identified.

3 Deformation theory of stable curves

Recall that in the functor viewpoint (that is, we assume to work on the stack M_g), $F_{\overline{M}_g}(B)$ should be equal to $\mathrm{Hom}(B, \overline{M}_g)$. When B is the spectrum of the dual numbers $\mathbb{C}[t]/t^2$, we get the tangent space at the point $[C] \in \overline{M}_g$, where $[C]$ is the fiber over (t). Associated to the map $B \to \overline{M}_g$ we have a curve $\pi\colon \mathcal{C} \to B$, and an exact sequence

$$0 \to \pi^\star \Omega^1_B|_C = \mathcal{O}_C \to \Omega^1_{\mathcal{C}}|_C \to \Omega^1_C \to 0\,,$$

because B is a disk. Taking the long exact sequence, we obtain that the tangent space is $\mathrm{Ext}^1(\Omega^2_C, \mathcal{O}_C) = \mathrm{Ext}^1(\Omega^1_C \otimes \omega_C, \omega_C) = \mathrm{H}^0(C, \omega_C \otimes \Omega_C)$, tensoring by ω_C and using Serre duality (recall that Ω^1_C is the sheaf of differential forms, whereas ω_C is the dualizing sheaf that is locally free).

Globalizing, we can take the universal cover $\mathcal{C}_g \to \overline{M}_g$ and the cotangent bundle will be $T^\vee_{\overline{M}_g} = \pi_\star(\Omega^1_\pi \otimes \omega_\pi)$.

Lecture 3 (1 hour) June 12th, 2012

Let C be a stable curve, with singularities $p_1, \dots p_\delta$, and let $i\colon \widetilde{C} \to C$ be the normalization, where $i^{-1}(p_i) = \{x_i, y_i\}$. Then $\omega_C = i_\star(\omega_{\widetilde{C}}(\sum x_i + y_i))$. Locally around a node, the Kähler differentials are not locally free: if the node is $xy = 0$, then $y(xdy) = 0$ (that implies that Ω^1_C is not locally free). On the other hand, ω_C is locally free, and at a node it is generated by dx/x (if $y = 0$), or viceversa.

We have then a local exact sequence

$$0 \to \mathrm{Tors}(\Omega^1_C) \to \Omega^1_C \to \omega_C \to \bigoplus_{p \in \mathrm{Sing}(C)} \mathbb{C}_p \to 0$$

(the last map is the residue map). We can globalize this sequence too: consider again the universal curve and let Σ be the set of all points $[C, p]$ of $\overline{\mathcal{C}}_g$ such that p is a node of C. It has codimension 2 in $\overline{\mathcal{C}}_g$. One can prove that $\Omega^1_\pi = \omega_\pi \otimes \mathcal{I}_\Sigma$, and so $\pi_\star(\omega_\pi^{\otimes 2} \otimes \mathcal{I}_\Sigma) = \Omega^1_{\overline{M}_g}$.

For any curve in $\overline{\mathcal{M}}_g$, we have that $\mathrm{H}^0(\omega_C \otimes \Omega_C)^\vee \,/\, \mathrm{Aut}(C)$ is a local model around $[C]$ for $\overline{\mathcal{M}}_g$.

4 On unirationality

4.1 theorem (Severi, 1915). *The space $\mathcal{M}_g$ is unirational for $g \leq 10$.*

Unirationality is an important property for a moduli space: it means that every curve can be described by some parameters and there are no "hidden" equations on these parameters. Severi conjectured that this was true for all genera, and this was a very believable conjecture.[1]

[1] Rationality is known to be true for $g \leq 6$.

To prove such a statement, one needs to construct a map from a rational variety to $\mathcal{M}_g$. Recall that it is defined

$$W^r_d(C) = \{L \in \operatorname{Pic}^d(C) \mid \mathrm{h}^0(L) \geq r+1\},$$

that for the general curve has dimension $\varrho \equiv (r+1)(g-d+r)$. For $r=2$, we obtain that the generic curve $[C] \in \mathcal{M}_g$ has a g^r_d where $d \geq (2g+16)/3$.

Now, take the embedding given by the g^r_d, that has a nodal image $\Gamma \subseteq \mathbb{P}^2$ with $\delta = \binom{d-1}{2} - g$ nodes. If one could prove that the nodes are in general position, then we are done. Consider the incidence correspondence

$$\Sigma = \{(\Gamma, p_1, \ldots, p_\delta) \mid \deg \Gamma = d, \{p_1, \ldots, p_\delta\} = \operatorname{Sing}(\Gamma)\}.$$

We have two maps $\pi_1 \colon \Sigma \to \mathcal{M}_g$ (the normalization) and $\pi_1 \colon \Sigma \to (\mathbb{P}^2)^d$. Fibers of π_2 are linear spaces, and if π_2 is dominant then Σ is unirational, and if $d \geq (2g-16)/3$ also $\mathcal{M}_g$ is unirational.

This can only happen if $\dim \Sigma \geq 2\delta$, but

$$\dim \Sigma = 3g-3+g-3(g-d+2) + \dim \mathbb{P}\,\mathrm{GL}(2+1) = 3d+g-1.$$

Putting all together, the three condition are satisfied only for $g \leq 10$.

Recent development improved further the result, even if the conjecture is known to be false.

4.2 theorem. *$\mathcal{M}_g$ is unirational for $g \leq 14$.*

A curve of genus 14 has a finite number of g^1_8 (a Catalan number). Let $L := K_C(-g^1_8) = g^6_{18}$: there are still finitely many L and we can use them to construct an embedding in $\mathbb{P}^6$. We look at quadrics containing the image, using $S^2\,\mathrm{H}^0(L) \to \mathrm{H}^0(L^2)$. Since the target has dimension $36+1-14 = 26$ and the source $\binom{8}{2} = 28$, the generic curve lies in exactly 5 quadrics. Let C' be the residual curve (that is, $\overline{\bigcap Q_i \setminus C}$): in general C' is a smooth curve of degree $2^5 - 18 = 14$. In particular C' is a non-special genus 8 curve with a g^6_{14}. We can go also the other way round: we take the incidence variety $\Sigma = \{(C', V^5)\}$ where C' is a genus 8 curve in $\mathbb{P}^6$ of degree 14 and $V^5 \in G(5, \mathrm{H}^0(\mathcal{I}_{C'}(2))$. So we have projections to $\operatorname{Pic}^{14}_8$ (the universal Picard variety over $\mathcal{M}_8$) and to $\mathcal{M}_{14}$ (reversing the construction of before). The first map is a Grassmann bundle and so if $\operatorname{Pic}^{14}_8$ is unirational, then also $\mathcal{M}_{14}$ would be. Mukai solved the question about $\operatorname{Pic}^{14}_8$, and so $\mathcal{M}_{14}$ is unirational.

Carrying along the same precedure for $g = 15$, we obtain that $\mathcal{M}_{15}$ is rationally connected. It is known then that $\mathcal{M}_{15}$ and $\mathcal{M}_{16}$ are uniruled, but $\mathcal{M}_{17}$ is completely open.

Lecture 4 (1 hour) June 13^{th}, 2012

We saw how we can compute the local structure of $\overline{\mathcal{M}}_g$ using the universal curve $\pi \colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$. We had two sheaves, the relative dualizing sheaf ω_C, which is locally free, and the Kähler differential sheaf, Ω^1_π, which is not locally free. Moreover, we have $\Omega_{\overline{\mathcal{M}}_g} = \pi_\star(\omega_\pi \otimes \Omega_\pi)$.

4.1 *Picard group of $\overline{\mathcal{M}}_g$*

Recall that we can see $\overline{\mathcal{M}}_g$ as $\mathcal{M}_g$ union Δ_0, the locus of nodal curves with one node, union Δ_i, the locus of curves with two components, of genera i and $g-i$, intersecting in a node, for $i \in \{1, \dots, \lfloor g/2 \rfloor\}$. Note that we consider the loci Δ_i as closed, that is, we include in them all their degenerations. All the strata Δ_i are in codimension 1, and we call δ_i their classes in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$. We need to consider the Picard group with rational coefficients, because, for example, the correct definition of δ_1 is $1/2[\Delta_1]$ (because of the involution in the elliptic curve).

Apart from these boundary classes, there are other "tautological" classes coming from the interior $\mathcal{M}_g$. We can define $\kappa_1 := \pi_\star(c_1(\omega_\pi)^2) \in \mathrm{H}^1(\overline{\mathcal{M}}_g)$, the Mumford κ class (one can define similar classes using higher powers). Finally, we define the Hodge bundle $\mathbb{E} := \pi_\star(\omega_\pi)$, of rank g, where $\mathbb{E}_{[C]} = \mathrm{H}^0(\omega_C)$. The λ class are $\lambda_1 = \lambda := c_1(\mathbb{E})$.

For genus at least 3, the Picard group of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is generated by the classes $\lambda_1, \delta_0, \dots, \delta_{\lfloor g/2 \rfloor}$. The κ class is not needed, indeed the Mumford's relation states that $\kappa_1 = 12\lambda - \delta$, where $\delta = \sum \delta_i$.

4.2 *Computing the canonical class of $\overline{\mathcal{M}}_g$*

We can use Grothendieck-Riemann-Roch to manipulate the identity $\Omega^1_{\overline{\mathcal{M}}_g} = \pi_\star(\omega_\pi \otimes \Omega^1_\pi)$: suppose we have a proper morphism $f\colon X \to Y$, and a sheaf J on X; we define $f_!\mathcal{F} := \sum_{i\geq 0} (-1)^i R^i f_\star \mathcal{F}$; then $\mathrm{ch}(f, \mathcal{F}) = f_\star(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(\Omega_f^\vee))$, where $\mathrm{td}(\Omega_f^\vee) = 1 - \frac{c_1(\Omega_f)}{2} + \frac{c_1^2(\Omega_f)+c_2(\Omega_f)}{12} + \cdots$.

In our case, $\mathcal{F} = \omega_\pi \otimes \Omega_\pi$, $R^1\pi_\star\mathcal{F} = 0$ for dimensional consideration (the stalk is a curve) and so $\pi_!\mathcal{F} = \pi_\star\mathcal{F}$. Also, $c_1(\omega_\pi) = c_1(\Omega_\pi)$, $c_2(\Omega_\pi) = [\Sigma] \in \mathrm{H}^2(\overline{\mathcal{C}}_g)$. Applying GRR, we obtain (recall that Σ is the locus of nodes, hence $\pi_\star(\Sigma) = \delta$)

$$
\begin{aligned}
k_{\overline{M}_g} &= c_1(\pi_\star\mathcal{F}) = \\
&= \pi_\star\Bigg[1 + c_1(\omega_\pi \otimes \Omega_\pi) + \frac{c_1^2(\omega_\pi \otimes \Omega_\pi) - 2c_2(\omega_\pi \otimes \Omega_\pi)}{2} \cdot \\
&\qquad \cdot \left(1 - \frac{c_1(\Omega_\pi)}{2} + \frac{c_1^2(\Omega_\pi) + c_2(\Omega_\pi)}{12}\right)\Bigg] = \\
&= -\pi_\star(c_1^2(\omega_\pi)) + \frac{\pi_\star(c_1^2(\omega_\pi))}{12} + \frac{\delta}{12} + 2\pi_\star(c_1^2(\omega_\pi)) = \\
&= \frac{13}{12}\kappa_1 - \frac{11}{12}\delta = \\
&= \frac{13}{12}(12\lambda - \delta) - \frac{11}{12}\delta = \\
&= 13\lambda - 2\delta\,.
\end{aligned}
$$

This computation was on the stack; for the space, we have a map $\overline{M}_g \to \overline{\mathcal{M}}_g$, branched along Δ_1, and we can compute the result using Hurwitz formula:

$$k_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor g/2 \rfloor} .$$

Using this computation, we can disprove the conjecture by Severi (that $\overline{\mathcal{M}}_g$ is unirational for all g) in the most spectacular way.

4.3 THEOREM (Harris-Mumford-Eisenbud). *The space $\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.*

To prove such a theorem, in theory one would proceed trying to prove that $k_{\overline{\mathcal{M}}_g}$ is effective (or big). There are serious issue on singularities of $\overline{\mathcal{M}}_g$, but we will ignore it since the main result is that these singularities does not impose adjunction conditions (by the Reid-Tai criterion).

The first fact is that the Hodge class λ is very positive, indeed is big and nef on $\overline{\mathcal{M}}_g$. This is already non-trivial and derives from the Torelli problem applied to stable curves (the map sends a curve to the Jacobian of its normalization in the Satake compactification of $\mathcal{A}_g$). One proves that this map t is regular, and that λ is the pullback via t of $\mathcal{O}(1)$, hence λ is big and nef.

Now we need to look if (and for which genera) λ is positive enough. The good thing to work on the moduli space of curves instead on a random variety is that we can define geometric sensible loci in $\overline{\mathcal{M}}_g$ using some property of curves that are not enjoyed by all curves, and if we have a good characterization we can also compute the class of these loci. For example, we have the loci $\mathcal{M}^r_{g,d} \subseteq \overline{\mathcal{M}}_g$ of curves admitting a g^r_d. If the Brill-Noether number $\varrho = (r+1)(g-d+r)$ is -1, then $\mathcal{M}^r_{g,d}$ is a divisor, and Eisenbud and Harris computed its class, that is

$$[\mathcal{M}^r_{g,d}] = c\left((g+3)\lambda + \frac{g+1}{6}\delta_0 - \sum i(g-i)\delta_i\right) .$$

Note that this formula does not depend on r and d.

4.4 EXAMPLE. For $g = 3$, $r = 1$, $d = 2$, for example, we can look at $\mathcal{M}^1_{3,2} = \mathcal{H}$, the locus of hyperelliptic curves. We can write $[\mathcal{H}] = a\lambda - b_0\delta_o - b_1\delta_1$, and we can compute the coefficients intersecting this locus with three other loci in $\overline{\mathcal{M}}_3$ of dimension 1. The first, C_0 is a curve of genus 2 with x identified with p and x moving on the curve, inside Δ_0; the second, C_1, is an elliptic curves intersecting with a genus two curve in a moving point, and this locus is in Δ_1; the third, R is the locus of a curve of genus 2 intersecting a pencil of cubics in a fixed point.

Consider C_1: its intersection with λ is zero because when we use the Torelli map, we forget about the marked point on the genus 2 curve and so the point does not move anymore. The intersection with δ_0 is 0 again, and the intersection with δ_1 is $-\deg k_C = -2$.

We can compute all the other intersections in similar ways, obtaining $[\mathcal{H}] = 9\lambda - \delta_0 - 3\delta_1$.

Lecture 5 (1 hour)
June 14th, 2012

Assume to have a divisor D in $\overline{\mathcal{M}}_g$, whose support does not intersect Δ_i; then we can write its class as $[D] = a\lambda - \sum b_i\delta_i$, with all coefficients non-negative. We define the slope of D as $s(D) = a/\min b_i \geq 0$: for example, $s(\lambda) = \infty$. We also define $s(\overline{\mathcal{M}}_g) := \inf s(D) \geq 0$ (where the infimum is taken over all effective divisors).

Note that whenever $s(\overline{\mathcal{M}}_g) < s(k_{\overline{\mathcal{M}}_g}) = 13/2$, $\overline{\mathcal{M}}_g$ is of general type; on the other hand, if $s(\overline{\mathcal{M}}_g) > 13/2$, then $\overline{\mathcal{M}}_g$ is uniruled. Indeed, in the first case we know that we can write $k_{\overline{\mathcal{M}}_g} = \alpha\lambda + \beta D + \Delta$, where Δ is some combination of boundary divisors; but then λ is big and nef, so in this case also $k_{\overline{\mathcal{M}}_g}$ is big and nef. In the other case, we can use the estimate on the slope to prove that the canonical divisor is not pseudo-effective and this solve the problem by a theorem of Bouckson, Demaille, Peternell, and Paun.

By the theorem of Harris-Mumford, we know that $s(\overline{\mathcal{M}}^r_{g,d}) = 6 + \frac{12}{g+1}$, that is less than $13/2$ for $g > 23$, that is, $\overline{\mathcal{M}}_g$ is of general type for $g > 23$. The problem is constructing a divisor of slope less than $13/2$, and they used the Brill-Noether divisor.

4.5 conjecture (Slope, Harris-Morrison, 1990). *The slope of $\overline{\mathcal{M}}_g$ is at least $6 + \frac{12}{g+1}$ with equality for $g+1$ composite and realized by the Brill-Noether divisor.*

If the conjecture were true, it would prove that $\overline{\mathcal{M}}_g$ is uniruled for $g < 23$.

5 Curves on K3 surfaces

Let S be a K3 surface with a curve C of genus g. Let $\{C_\lambda\}_{\lambda\in\mathbb{P}^1}$ be a Lefschetz pencil, that is, if we take two curves $C_1, C_2 \subseteq S$ in the pencil, then they intersect in $2g-2$ points. We blow up these points, obtaining a fibration over $\mathbb{P}^1$. This construction induces a $\mathbb{P}^1$ inside $\overline{\mathcal{M}}_g$.

The intersection $R\cdot\lambda$ is $\chi(\hat{S}, \mathcal{O}_{\hat{S}}) + g - 1 = g + 1$, while $R\cdot\delta_i = 0$ for $i > 0$, and so $R\cdot\delta_0 = R\cdot\delta$. But then we have a formula for the Euler characteristic of the fibration: $e(\hat{S}) = e(\mathbb{P}^1)e(F) + R\cdot\delta$, therefore $R\cdot\delta_0 = R\cdot\delta = 6(g+3)$.

We obtain in particular that $\frac{R\cdot\delta}{R\cdot\lambda} = 6 + \frac{12}{g+1} = s(\overline{\mathcal{M}}^r_{g,d})$.

Now, consider the moduli space $\mathcal{F}_g$ of K3 surfaces polarized of degree $h^2 = 2g-2$, of dimension 19; then there is $\mathcal{P}_g$, the moduli space of K3 surfaces together with a curve in $|h|$, of dimension g. This latter space have a map π_1 to the former (forgetting the curve) and π_2 to $\mathcal{M}_g$ (forgetting the surface). In particular, $\pi_2(\mathcal{P}_g)$ is the locus k_g of curves on K3 surfaces.

5.1 proposition (Farkas-Popa). *If $D \subseteq \overline{\mathcal{M}}_g$ is an effective divisor of slope $6 + \frac{12}{g+1}$, then $D \supseteq k_g$.*

If the slope conjecture were true, then this statement would be empty. In particular, for $g \leq 9$ and $g = 11$, we have $k_g = \mathcal{M}_g$, hence the slope conjecture is true. In genus 10, the map π_2 has fiber dimension equal to 3 (Mukai), hence k_{10} is a divisor and the only possible counter-example to the slope conjecture in this genus. And indeed, it is a counterexample, because computing the class gives $[k_{10}] = 7\lambda - \delta_0 - 5\delta_1 - 9\delta_2 - 12\delta_3 - 14\delta_4 - 15\delta_5$, so the slope is 7.

5.2 theorem (Farkas-Popa). *The slope conjecture is false in genus* 10.

6 Koszul cohomology

Lecture 6 (1 hour)
June 15th, 2012

We have constructed a divisor k_{10}, called the Koszul divisor, in $\mathcal{M}_{10}$, with slope less than $6 + 12/(g+1)$. How can one generalize such a divisor? The answer is syzygies, and Koszul cohomology. Consider a projectively normal curve C, L a line bundle with corresponding embedding $C \to \mathbb{P}\,\mathrm{H}^0(L)^\vee =: \mathbb{P}^\vee$; then we compute a minimal free graded resolution

$$\cdots \to F_2 \to F_1 \to I_c \to 0\,,$$

so that $F_i = \bigoplus_{j\in\sum} S(-i-j)^{b_{i,j}}$. The Koszul cohomology can compute the $b_{i,j} = b_{i,j}(C,L) = \dim_{\mathbb{C}} \mathrm{Tor}^i(S(C,L),\mathbb{C})$, the i-th order syzygies of degree j. The Koszul cohomology is constructed from the sequence

$$\bigwedge^{i+1} \mathrm{H}^0(L)\otimes \mathrm{H}^0(L^{j-1}) \xrightarrow{d_{i-1,j-1}} \bigwedge^{i} \mathrm{H}^0(L)\otimes \mathrm{H}^0(L^{j}) \xrightarrow{d_{i,j}}$$
$$\xrightarrow{d_{i,j}} \bigwedge^{i-1} \mathrm{H}^0(L)\otimes \mathrm{H}^0(L^{j+1}) \to \cdots\,,$$

where $d_{i,j}(f_1\wedge\cdots f_i\otimes s) = \sum_l (-1)^i f_1\wedge\cdots \hat{f_l}\cdots\wedge f_i\otimes f_l s$. The Koszul cohomology is then $k_{i,j}(C,L) = \ker d_{i,j}/\operatorname{Im} d_{i+1,j+1}$. Indeed, $b_{i,j} = \dim k_{i,j}$.

We can stratify $\mathcal{M}_g$ using syzygies. We fix integers g,r,d such that $\varrho = g-(r+1)s = 0$, where $s = (g-d+r)$ (in particular, $g = rs+s$). We define the space $\mathcal{G}^r_d = \{(C,L) \mid L = g^r_d\}$, that has a projection σ to $\mathcal{M}_g$, and σ is a generically finite covering. We define also the subspace $\mathcal{Z}_{g,i} := \{(C,L) \in \mathcal{G}^r_d \mid k_{i,2}(C,L) \neq 0\}$, that is, the space of couples for which we have non-linear syzygies of order i. We want to consider $[\sigma_\star|_{\mathcal{Z}_{g,i}}] \in \mathrm{H}^\star(\mathcal{M}_g)$.

We can expect this element to be a divisor on $\mathcal{M}_g$ when $\mathrm{H}^0(\Omega^i_{\mathbb{P}^r}(i+2)) \to \mathrm{H}^0(\Omega^0_{\mathbb{P}^r}(i+2)|_C)$ is a map between vector spaces of the same rank (in this case, the rank is $r = 2s+si+i$). The Lazersfeld bundle of (C,L) is the rank r vector bundle M_L fitting in the exact sequence

$$0 \to M_L \to \mathrm{H}^0(L)\otimes\mathcal{O}_C \xrightarrow{\mathrm{ev}} K \to 0\,;$$

using it we can reformulate the map as $\mathrm{H}^0(\bigwedge^i M_{\mathbb{P}^r}(2)) \to \mathrm{H}^0(\bigwedge^i M_L\otimes L^i)$, since $M_{\mathbb{P}^r} = \Omega_{\mathbb{P}^r}(i)$; so we can reformulate $\mathcal{Z}_{g,i}$ as the locus in $\mathcal{G}^r_d$ of (C,L) such that that map is not an isomorphism.

Summing up, taking $s,i\geq 0$, and putting $r = 2s+si+i$, $g = rs+s$, then $\mathcal{U}_{g,i} = \sigma_\star(\mathcal{Z}_{g,i}) = \{[C]\in\mathcal{M}_g \mid \exists L = g^r_d : k_{i,2}(C,L)\neq 0\}$ is a virtual divisor on $\mathcal{M}_g$.

6.1 example. If $i = 0$, then $r = 2s$, $g = s(2s+1)$, $d = 2s(s+1)$; then $\mathcal{Z}_{g,0} = \{(C,L)\in\mathcal{G}^r_d \mid S^2\,\mathrm{H}^0(L) \xrightarrow{\not\sim} \mathrm{H}^0(2L)\}$. The first space has dimension $\binom{r+2}{2}$, the second $2d+1-g$. In the case $s=2$, $i=0$, $g=10$, then $\sigma_\star(\mathcal{Z}_{10,0})$ is the space of curves such that there exists a g^4_{13} that maps the curve inside a quadric in $\mathbb{P}^4$, and as a divisor is equal to $42k_{10}$.

"Virtual" divisor means that we have good reasons to hope that it is a divisor, but there is no guarantee that it will be an actual divisor. In the examples, it is not hard to prove that the constructed subvarieties are honest divisors, but in general it is hard, even if it is enough to prove that a single curve does not belong to the locus.

6.2 example. If $s=1$, then $r=2i+2$, $d=4i+6=2g-2$, $g=2i+3$. In this case, there is only one g^{g-1}_{2g-2}, the canonical bundle of C, hence in this case $\mathcal{G}^r_d \to \mathcal{M}_g$ is the identity. The locus $\mathcal{Z}_{g,i}$ of curves in $\mathcal{M}_{2i+3}$ such that $k_{1,2}(C,k_C) \neq 0$ has attracted a lot of attention in the last years because of the Green's conjecture, stating that for every curve C, we have $k_{p,2}(C,k_C)=0$ if and only if $p < \operatorname{Cliff}(C)$, and $\operatorname{Cliff}(C) = \lfloor (g-1)/2 \rfloor = i+1$ for the generic curve. If the conjecture holds, then $\mathcal{Z}_{g,i}$ is, as a set, the locus of curves with a g^i_{i+2}, that is, $\mathcal{M}^1_{g,i+2}$. Remarkably, also the converse is true, so $\mathcal{Z}_{g,i} \neq \mathcal{M}_{2i+3}$ is equivalent to Green's conjecture for generic curves. This is a result of Voisin.

6.3 theorem. *There is an explicit expression for the virtual class* $[\sigma_\star(\mathcal{Z}_{g,i})] \in \mathrm{H}^i(\overline{\mathcal{M}}_g)$, *and its slope is always less or equal than* $6+12/(g+1)$, *with strict inequality for* $s \geq 2$ *and equality for* $s=1$.

6.4 example. Though the expression is quite long, in some cases it is nice: for $s=2$, $r=3i+4$, $d=9i+12$, $g=6i+10$, $L=k_C(-g^1_{3i+6})$, we have that the slope is

$$s(\sigma_\star(\mathcal{Z}_{i,0})) = \frac{3(4i+7)(6i^2+19i+12)}{(i+2)(2i^2+31i+18)} .$$

In the case $g=22$, $\mathcal{Z}_{22}$ is the locus of curves admitting a g^{10}_{30} with $k_{2,2} \neq 0$. This is again in contrast with the slope conjecture, because the slope of this divisor is $1665/256 = 6.5003\ldots < 6 + 12/23$. But this is still not enough to prove that $\mathcal{M}_{22}$ is of general type: recall that $\mathcal{M}_g$ is of general type if and only if there exists $D \subseteq \mathcal{M}_g$ with slope less than $13/2$.

It is known that Koszul divisors are divisors for $s=1$ and $i=0$. In general, the slope of the Koszul divisors always lie in the range $(6+10/g, 6+12/g+1)$: the left endpoint is the slope of a pencil of 1-nodal curves on a K3 surface.

Lecture 7 (1 hour)
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But, there is another candidate that succeed in proving that $\mathcal{M}_{22}$ is of general type. We consider $\varrho = 1$ and in particular linear systems g^6_{25}. There are ∞^1 of these systems, and $\mathcal{G}^6_{25} = \{(C,L) \mid g(C)=22, L = g^6_{25} = k_C(-g^2_{17})\}$. The fibers over $\mathcal{M}_{22}$ are one dimensional, called $W^6_{25}(C)$. On $\mathcal{G}^6_{25}$ one defines two tautological bundles: $\mathcal{E} := S^2\mathrm{H}^0(L)$ and $\mathcal{F} := \mathrm{H}^0(L^2)$. There is a morphism $\chi\colon \mathcal{E} \to \mathcal{F}$ given by the multiplication of sections and $\operatorname{rk}\mathcal{E} = 28$, while $\operatorname{rk}\mathcal{F} = 2\deg L + 1 - g = 29$ (by Riemann-Roch). Hence $\mathcal{E} \to \mathcal{F}$ is not injective, and we can look at the locus where the rank is 27, that is, $\mathcal{Z} = \{(C,L) \mid \chi(\mathcal{E},\mathcal{F}) \text{ non-injective}\}$. Another interpretation is that $\mathcal{Z}$ is the

$\lim_{g\to\infty} s(\overline{\mathcal{M}}_g) = 0$	$\lim_{g\to\infty} s(\overline{\mathcal{M}}_g) > 0$
Morrison, Chen, Pandharipande	Farkas

Table 1: The bet on the limit of the slope.

locus of (C, L) such that there exists a g^6_{25} such that the embedded curve lies on a quadric. This has expected codimension 2 in $\mathcal{G}^6_{25}$, so the image $\sigma_\star(\mathcal{Z})$ should be a divisor in $\mathcal{M}_{22}$.

6.5 THEOREM. *The subvariety $\sigma_\star(\mathcal{Z})$ is indeed a divisor on $\mathcal{M}_{22}$, that is, a general curve C does not lie on a quadric in* any *of the ∞^1 embeddings g^6_{25}. Moreover, $s(\sigma_\star(\mathcal{Z})) = 17121/2636 = 6.4956\ldots$, hence $\mathcal{M}_{22}$ is of general type.*

Another interesting question is what is the asymptotic behaviour of $s(\overline{\mathcal{M}}_g)$ for $g \to \infty$. This would be interesting for the Schottky problem, that is, finding a characterization of the image of the Torelli map $t\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{A}}_g$, where the compactification $\overline{\mathcal{A}}_g$ is such that the boundary is irreducible. Tai proved that the limit of the slope of $\overline{\mathcal{A}}_g$ is 0, and one could ask if this could be pulled back on $\overline{\mathcal{M}}_g$. In this case one could hope to find another solution of the Schottky problem.

We already saw that $s(\overline{\mathcal{M}}_g) < 6 + 12/(g+1)$ for $g \gg 0$. There has been an attempt to find a lower bound by Harris-Morrison: they looked for a curve $Y \subseteq \overline{\mathcal{M}}_g$ that is sweeping the space; then every effective divisor $D \subseteq \overline{\mathcal{M}}_g$ must intersect the curve non-negatively. If $[D] = a\lambda - b\delta$, this means that $a\lambda \cdot Y \geq b\delta \cdot Y$, hence $s(D) \geq \delta \cdot Y/\lambda \cdot Y$ and $s(\overline{\mathcal{M}}_g) \geq \delta \cdot Y/\lambda \cdot Y$. The problem is that it is very hard to find such curves.

A possibility is to consider the space $\mathcal{H}_{g,h}$ of $k : 1$ covers of $\mathbb{P}^1$ with $b = 2g + 2k - 2$ simple branch points. This has two projections, π_1 to $\overline{\mathcal{M}}_g$ and π_2 to $\overline{\mathcal{M}}_{0,8}$. Letting p_b move in $\mathbb{P}^1$ one get a moving curve $F \subseteq \overline{\mathcal{M}}_{0,8}$, and so a moving curve $(\pi_1)_\star \pi_2^\star(F)$ on $\overline{\mathcal{M}}_g$. The computations here are very heavy. The term $\delta \cdot Y/\lambda \cdot Y$ can be computed using the characters of S_k, and it can be done for any finite genus. The heuristic says that its limit for $g \to \infty$ is $576/5g = O(1)$, so the slope of $\overline{\mathcal{M}}_g$ should be at least $576/5g$, which is not so very satisfactory, being so far from $6 + 12/(g+1)$.

Another approach was done by Dawa-Chen using Teichmüller curves $T_{g,t}$, that are analytic curves in $\overline{\mathcal{M}}_g$ still defined in terms of covers. These curves are rigid in $\overline{\mathcal{M}}_g$, but there are many of them and their union is dense in $\overline{\mathcal{M}}_g$. So, $s(\overline{\mathcal{M}}_g) \geq \inf_t T_{g,t} \cdot \delta / T_{g,t} \cdot \lambda$, and the interesting thing is that the limit of this infimum is (heuristically) the same: $576/5g$. This could be a justified by the fact that both approaches uses covers.

The third attempt, from Pandharipande, uses the same approach of Harris-Morrison but in a simpler way: he construct a moving curve as a complete intersection of numerically ample effective divisors: he took the universal curve $\pi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ and the universal cotangent bundle ψ, which is nef. Then, $\pi_\star(\psi^{3g-3})$ is a moving curve, and we have that the slope of $\overline{\mathcal{M}}_g$ is at least $60/(g+4)$ and this is not an heuristic but an actual result.